# CALCULUS I

**BAMAT-101**

**Self Learning Material**

**Directorate of Distance Education**

**SWAMI VIVEKANAND SUBHARTI UNIVERSITY**

**MEERUT-250005**

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**BA (Match)**

**SEMESTER I**

**Course I**

**Course Name:** Calculus I **Course Code:** BAMAT-101

| | |
|---|---|
| **Course Objectives:** | It is a basic course on the study of real valued functions that would develop an analytical ability to have a more matured perspective of the key concepts of calculus, namely, limits, continuity, differentiability and their applications. |
| **Unit 1:** | Functions, types, domain and range, Limits of functions, Sequential criterion for limits, Divergence criteria, Limit theorems, One-sided limits, Infinite limits and limits at infinity. |
| **Unit 2:** | Continuous functions, Sequential criterion for continuity and discontinuity, Algebra of continuous functions, Properties of continuous functions on closed and bounded intervals; Uniform continuity, Non-uniform continuity criteria, Uniform continuity theorem. |
| **Unit 3:** | Differentiability of functions, Successive differentiation, Leibnitz's theorem, Partial differentiation, Jacobians, Euler's theorem on homogeneous functions , proof and applications. |
| **Unit 4:** | Tangents and normals, Envelopes and Evolutes, Curvature, Asymptotes, Singular points. Curve tracing |

**Course Learning Outcomes:** This course will enable the students to learn:

1. To have a rigorous understanding of the concept of limit of a function.
2. The geometrical properties of continuous functions on closed and bounded intervals.
3. The applications of mean value theorem and Taylor's theorem.

**Reference:**


1. Bartle, Robert G., & Sherbert, Donald R. (2015). *Introduction to Real Analysis* (4th ed.). Wiley India Edition. New Delhi.


**Additional Readings:**


1. Ghorpade, Sudhir R. & Limaye, B. V. (2006). *A Course in Calculus and Real Analysis*. Undergraduate Texts in Mathematics, Springer (SIE). First Indian reprint.
2. Mattuck, Arthur. (1999). *Introduction to Analysis*, Prentice Hall.
3. Ross, Kenneth A. (2013). *Elementary Analysis*: *The theory of calculus* (2nd ed.). Undergraduate Texts in Mathematics, Springer. Indian Reprint.

# UNIT I

# FUNCTIONS

**STRUCTURE**



## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Types of Fuctions
- Operations on Real Functions
- Composition of Real Functions
- Inverse of a Real Function

## INTRODUCTION

Differential calculus deals with the problem of calculating rates of change. The 'function' concept lays the foundation of the study of the most important branch calculus of mathematics. The word 'function' is derived from a Latin word meaning 'operation'. In this chapter, we study some frequently used common real valued functions and we shall study the properties of some of the most basic functions.

## DEFINITION

**Constant.** A quantity which have the same value throughout a mathematical operation is called a constant. *e.g.,* $2, -7, \sqrt{3}, \pi$ etc.

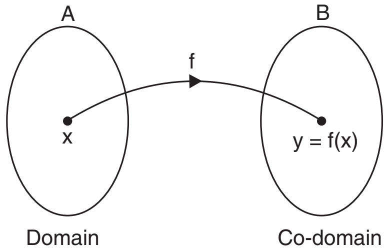
**Variable.** A quantity which can assume different values in a particular problem is called a variable.

*e.g.,* If $x$ represents any number between 3 and 7, then $x$ is a variable.

## FUNCTION

NOTES

Let A and B be two non-empty sets of real numbers. If there exists a rule '$f$' which associates to every element $x \in$ A, a unique element $y \in$ B, then such a rule $f$ is called a function (or mapping) from the set A to the set B.




If $f$ is a function from A to B, then we write :

$$f : A \rightarrow B.$$

The set A is called the domain of function $f$, and the set B is called the codomain of $f$.

If $x$ is an element of set A, then the element in B that is associated to $x$ by $f$ is denoted by $f(x)$ and is known as the image of $x$ under $f$ or the value of $f$ at $x$, and we write $f(x) = y$.

If $f(x) = y$, then we also say that $x$ is a pre-image of $y$.

The variables $x$ and $y$ are respectively called the independent variable and the dependent variable of the function. This is so, because each $y$-value depend on the corresponding $x$-value.

A function of $x$ is generally denoted by the symbol $f(x)$ and read as "$f$ of $x$".

**Caution.** $f(x) \neq f \times x$.

### Range

The range of a function $f : A \rightarrow B$ is the set of all those element of B which are having their pre-images in set A.

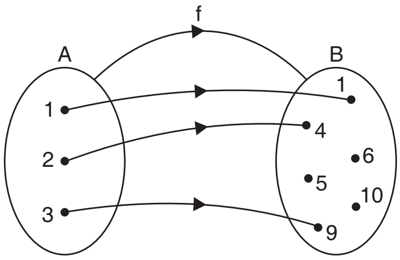
Or the range of a function is the set of images of elements of its domain.

*i.e.,* Range of $f = \{ f(x) : x \in A\}$.

Range $(f) \subseteq$ co-domain of $f$.

**Illustration.** Let A = {1, 2, 3},

B = {1, 4, 5, 9, 10}




Let $f : A \rightarrow B$ be the mapping which assigns to each element in A, its square.

Thus, we have $f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$.

Since to each element (1 or 2 or 3) of A, there is exactly one element of B, so $f$ is a function. In this case every element of B is not image of some element of A.

$\therefore$ We have,

Domain = {1, 2, 3}

Co-domain = {1, 4, 5, 9, 10}

Range = {1, 4, 9}.

### Real Valued Functions (Real Functions)

A function $f : A \rightarrow B$ is said to be a real function if and only if both A and B are the subsets of the real number system R.

*e.g.,* The function $g : R \rightarrow R$ defined by $g(x) = x^2 + 1$. $\forall x \in R$, is a real function.

A real function is generally described only by a formula and the domain of the function is not explicitly stated.

In such cases, the domain of the function is the set of all those real numbers $x$ for which the function $f(x)$ is meaningful.

*i.e.,* $f(x) \in R$, as the domain of $f$.

*e.g.,* we have $$f(x) = \sqrt{x - 5}$$

Here, $f(x)$ is defined if $x - 5 \geq 0$ *i.e.,* $x \geq 5$

$\therefore$ Domain $(f) = \{x : x - 5 \geq 0\} = \{x : x - 5 \geq 0, x \in R\}$.

If $f(a)$ is any of the forms $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$.

Then, we say that $f(x)$ is not defined at $x = a$.

NOTES

### Equal Functions

Two functions are said to be equal (coincide) if their domains, of definition coincide and their values for all identical values of the arguments are equal.

In other words :

$$f, g : A \rightarrow B \text{ are equal if } f(x) = g(x), x \in A.$$

**Illustration.** The function $f(x) = 2$ and $g(x) = 1 + \sin^2 x + \cos^2 x$ coincide.

**Illustration.** Let $f(x) = \frac{x^2 - 25}{x - 5}$, $x \in R - \{5\}$ and $g(x) = x + 5$, $x \in R$.

The functions $f$ and $g$ are not equal because $f$ is not defined at 5 whereas $g$ is defined at 5 and has value 10 there at.

Here, we note that $f(x) = g(x)$ for $x \in R - \{5\}$.

## TYPE OF FUNCTIONS

**1. One-one function (or Injective mapping).** A function $f : A \rightarrow B$ is said to be a one-one function, if the images of distinct elements of A are also distinct elements of B.

*i.e.,* $$x \neq y \Rightarrow f(x) \neq f(y)$$

Equivalently, $f(x) = f(y) \Rightarrow x = y \quad \forall x, y \in A$.

*e.g.,* $f : A \rightarrow B$ be the function defined by

$f(x) = 2x + 5$ is one-one function.

**2. Many one function.** A function $f : A \rightarrow B$ is said to be many one function if two or more elements of set A have the same image in B.

*e.g.,* $f : R \rightarrow R$ defined by

$f(x) = x^2$ is many one function.

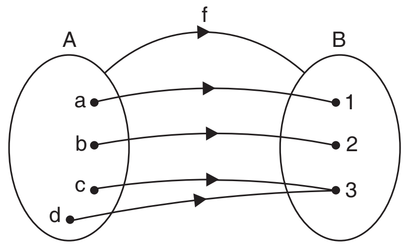
$\because$ $f(-2) = f(2) = 4$.

**3. Onto function. (or Surjective mapping).** A function $f : A \rightarrow B$ is said to be an onto function if every element $y \in B$, has at least one pre-image $x \in A$.

**NOTES**

In other words, for $y \in$ B, there exists at least one $x \in$ A such that :

$$f(x) = y$$

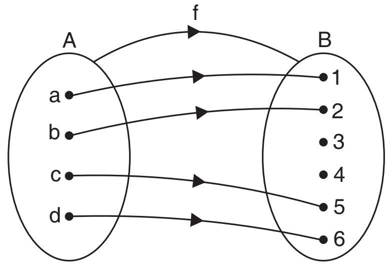



*e.g.,* $f : R \to R$ defined by :

$$f(x) = 4x + 5 \text{ is an onto function.}$$

**Note.** If the function $f$ : A → B is onto, then range of $f$ *i.e.,* the image of A is whole B.

**4. Into function.** A function $f$ : A → B is said to be an into function if for at least one $y \in$ B, which has no pre-image in the set A.




The adjoining diagram illustrates an into function,

because 3, 4 ∈ B has no pre-image in A.

*e.g.,* The function

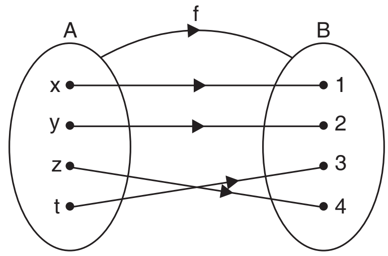
$f : R \to R$ defined by

$$f(x) = x^2$$

is an into function, because there is no real number whose image is a negative real number.

**Note.** A function which is not onto is called an into function.

**5. One-one onto function. (or Bijection function).** A function $f$ : A → B is said to be one-one onto function if it is both one-one and onto.




*e.g.,* The function

$f$ : A → B defined by

$$f(x) = 5x + 3,\ x \in A$$

is an one-one onto function.

**Note.** Let A and B are finite sets, and $f$ : A→B is a function :
Then,

(*i*) If $f$ is one-one, then $n(A) \leq n(B)$

(*ii*) If $f$ is onto, then $n(A) \geq n(B)$.

(*iii*) If $f$ is both one-one and onto, then $n(A) = n(B)$.

**6. Even function.** A function $f$ is even if $f(-x) = f(x)$ for all values of $x$.

*e.g.,* $f(x) = \cos x$ is an even function.

because, $f(x) = \cos x$

and $f(-x) = \cos(-x) = \cos x = f(x)$.

**7. Odd function.** A function $f$ is odd if $f(-x) = -f(x)$ for all values of $x$.

*e.g.,* $f(x) = \sin x$ is an odd function.

because, $f(x) = \sin x$

and $f(-x) = \sin(-x) = -\sin x = -f(x)$.

$[\because\ f(-x) = -x + 1 \neq f(x) \text{ and } f(-x) \neq -f(x)]$

**Note.** Every function need not be even or odd. *e.g.,* The function $f(x) = x + 1$ is neither even nor odd.

**8. Periodic function.** A function $f(x) = y$ is said to be a periodic function if there exists a real number $a > 0$ such that :

$$f(x + a) = f(x)$$

Then, $a$ is called period of the function.

*e.g.,* sin $x$, cos $x$, sec $x$ and cosec $x$ are periodic functions with period $2\pi$, while tan $x$ and cot $x$ are periodic with period $\pi$.

**9. Identity function.** The function $f : A \to B$ defined by $f(x) = x$ *i.e.,* each element of the set A is associated onto itself, then the function $f$ is called an identity function.

**10. Inverse function.** If a function $f : A \to B$ is one-one and onto function.

$\therefore$ for each $y \in B$, there exists unique $x \in A$ such that :

$$f(x) = y$$

Then, we can define an inverse function, which is denoted by $f^{-1} : B \to A$ and $f^{-1}(y) = x$ if and only if $f(x) = y$.

**Note.** (*i*) Every function does not have inverse. A function has inverse if and only if it is one-one and onto.

Also, $f(x) = y \Leftrightarrow x = f^{-1}(y)$.

(*ii*) $f^{-1}$ if it exists is unique.

(*iii*) The inverse of the identity function is the identity function itself.

**11. Composite function.** Let $f : A \to B$ and $g : B \to C$ be two functions.

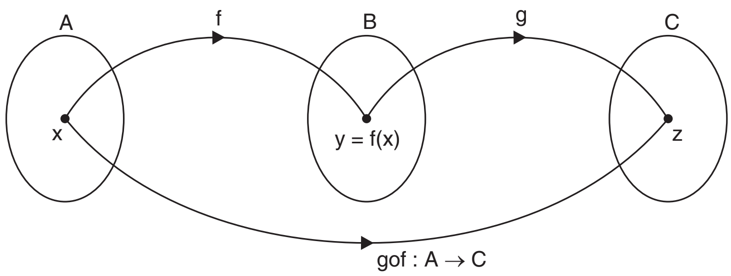
$\therefore$ for each $x \in A$, there exists a unique element $f(x) \in B$.

Since, $g : B \to C$ is a function, so $g(f(x))$ is a unique element of C.

Thus, to each $x \in A$, there exists exactly one element $g(f(x))$ in C. This correspondence between the elements of A and C is called the composite function of $f$ and $g$ is denoted by *gof*.

$$\therefore \quad (gof)(x) = g(f(x)),\ x \in A.$$

The composite function can be represented by the following diagrams.




**Note.** (*i*) *gof* is composite of $f$ and $g$ whereas *fog* is composite of $g$ and $f$.

(*ii*) In general $fog \neq gof$.

(*iii*) The existence of *fog* and *gof* is independent of each other. *i.e.,* if *fog* exists then *gof* may or may not exists and vice-versa.

**Illustration.** Let $f : R \to R$ and $g : R \to R$ defined by

$$f(x) = 4x + 5 \text{ and } g(x) = 5x - 2.$$

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$\therefore \quad (gof)\, x = g(f(x)) = g(4x + 5)$

$= 5(4x + 5) - 2 = 20x + 25 - 2 = 20x + 23$

and $\quad (fog)\, x = f(g(x)) = f(5x - 2)$

$= 4(5x - 2) + 5 = 20x - 8 + 5 = 20x - 3.$

Now, let us find $(fog)$ (5) ;

$\therefore \quad (fog)\,(5) = f(g(5)) = f(5(5) - 2) = f(23) = 4(23) + 5 = 92 + 5 = 97.$

Also, $\quad (fog)\,(5) = 20(5) - 3 = 100 - 3 = 97. \quad [\because \; (fog)\,(x) = (20x - 3)]$

## OPERATIONS ON REAL FUNCTIONS

In this section, we shall learn how to add two real functions, subtract a real function from another, multiply a real function by a scalar or another real function, divide one real function by another.

(*i*) **Addition of two real functions.** Let $f$ and $g$ be two real functions whose domains are $D(f)$ and $D(g)$ respectively. Let $D = D(f) \cap D(g) \neq \phi$.

Then, their sum $(f + g)$ is a function defined by :

$$(f + g)\,(x) = f(x) + g(x) \text{ for all } x \in D.$$

(*ii*) **Subtraction of a real function from another.** Let $f$ and $g$ be two real functions whose domains are $D(f)$ and $D(g)$ respectively. Let $D = D(f) \cap D(g) \neq \phi$.

Then, difference of $g$ from $f$ is denoted by $(f - g)$ and is defined as :

$$(f - g)\,(x) = f(x) - g(x) \text{ for all } x \in D.$$

(*iii*) **Multiplication of a function by a scalar.** Let $f$ be a real function whose domain is $D(f)$ and $\alpha$ be any scalar. Here, by scalar we mean a real number.

Then, the product $\alpha f$ is a function from $D(f)$ to R defined by :

$$(\alpha f)\,(x) = \alpha f(x) \text{ for all } x \in D.$$

(*iv*) **Multiplication of two real functions.** Let $f$ and $g$ be two real functions whose domains are $D(f)$ and $D(g)$ respectively. Let $D = D(f) \cap D(g) \neq \phi$.

Then, their product (or pointwise multiplication) $fg$ is a function defined by :

$$(fg)\,(x) = f(x)\, g(x) \text{ for all } x \in D.$$

(*v*) **Quotient of two real functions.** Let $f$ and $g$ be two real functions whose domains are $D(f)$ and $D(g)$ respectively. Let $D = D(f) \cap D(g) \neq \phi$.

Then, the quotient of $f$ by $g$ is denoted by $\dfrac{f}{g}$ and is defined by :

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for all } x \in D(f) \cap D(g) - \{x : g(x) = 0\}.$$

**Remark.** (*i*) The sum $(f + g)$, the difference $(f - g)$ and the product $fg$ are defined only when $f$ and $g$ are real functions having the same domain. In case $f$ and $g$ have different domains, one can define these operations for those points $x$ that are common to the domains of both $f$ and $g$ *i.e,* the points, which are in the intersection of the domains of $f$ and $g$.

(*ii*) Domain of $\dfrac{f}{g}$ is D – the set of those points where $g(x) = 0$.

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## COMPOSITION OF REAL FUNCTIONS

Let $f$ and $g$ be two real functions whose domains are D($f$) and D($g$) respectively and the domain of $f$ includes the range of $g$.

Let $$D = \{x : x \in D(g) \text{ and } g(x) \in D(f)\} \neq \phi.$$

Then, the composite of $f$ and $g$, denoted by $fog$ is a function defined by :

$$(fog)(x) = f(g(x)) \text{ for all } x \in D.$$

**Note.** (*i*) If $R(g) \cap D(f) = \phi$,

Then, $D = \phi$ and $fog$ is not defined.

(*ii*) If $R(g) \subset D(f)$,

Then, $D = D(g)$.

$\therefore$ The composite of $g$ and $f$, denoted by $gof$ is a function defined by :

$$(gof)(x) = g(f(x)) \text{ for all } x \in D$$

where $$D = \{x : x \in D(f) \text{ and } f(x) \in D(g)\} \neq \phi.$$

(*iii*) If $R(f) \cap D(g) = \phi$,

Then, $D = \phi$ and $gof$ is not defined.

Hence, if $R(f) \cap D(g) = \phi$, then $gof$ does not exists. In other words, $gof$ exists if

$$R(f) \cap D(g) \neq \phi.$$

Similarly, $fog$ exists if $R(g) \cap D(f) \neq \phi$.

## INVERSE OF A REAL FUNCTION

Let $f : X \to Y$ be a one-one and onto function.

$\therefore$ For each $y \in Y$, there exists a unique $x \in X$ such that $f(x) = y$.

$\therefore$ We get a function, denoted by $f^{-1}$ and defined as :

$$f^{-1} : Y \to X$$

such that $f^{-1}(y) = x$ iff $f(x) = y$

The function $f^{-1}$ is called the inverse function of $f$.

Clearly,

$$f^{-1}(f(x)) = f^{-1}(y) = x$$
$$f(f^{-1}(y)) = f(x) = y.$$

**Remark.** (*i*) Every function does not have inverse. A function has inverse if and only if it is one-one and onto.

(*ii*) The inverse of the identity function is the identity function itself.

(*iii*) Note that the function $f^{-1}$ and $\frac{1}{f}$ for any $f$, need not be same.

### SOLVED EXAMPLES

**Example 1.** *If* $f(x) = \frac{x}{x^2+1}$, *find* $f(1), f(3), f\left(\frac{2}{x}\right)$ *and* $f(b)$.

**Solution.** We have, $$f(x) = \frac{x}{x^2+1}$$

NOTES

$$\therefore \quad f(1) = \frac{1}{1^2+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$f(3) = \frac{3}{(3)^2+1} = \frac{3}{9+1} = \frac{3}{10}$$

$$f\left(\frac{2}{x}\right) = \frac{2/x}{\left(\frac{2}{x}\right)^2+1} = \frac{2/x}{\frac{4}{x^2}+1} = \frac{2/x}{\frac{4+x^2}{x^2}} = \frac{2}{x}\cdot\frac{x^2}{4+x^2} = \frac{2x}{x^2+4}$$

$$f(b) = \frac{b}{b^2+1}.$$

**Example 2.** (*i*) *If* $f(x) = \frac{x-1}{x+1}$, *then show that* $f\left(\frac{1}{x}\right) = -f(x)$

(*ii*) *If* $f(x) = x + \frac{1}{x}$, *then prove that* $[f(x)]^3 = f(x^3) + 3\, f\left(\frac{1}{x}\right)$.

(*iii*) *If* $f(x) = \frac{16^x}{16^x+4}$, *then show that* $f(x) + f(1-x) = 1$.

**Solution.** (*i*) We have,

$$f(x) = \frac{x-1}{x+1}$$

$$\therefore \quad f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = \frac{1-x}{1+x} = -\left(\frac{x-1}{x+1}\right) = -f(x).$$

(*ii*) We have, $\quad f(x) = x + \frac{1}{x}$ ...(1)

$$\therefore \quad [f(x)]^3 = \left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right) \quad ...(2)$$

Now, $\quad f(x) = x + \frac{1}{x}$

$$\therefore \quad f(x^3) = x^3 + \frac{1}{x^3} \quad ...(3)$$

and $\quad f\left(\frac{1}{x}\right) = \frac{1}{x} + \frac{1}{\frac{1}{x}} = \frac{1}{x} + x = f(x)$ ...(4)

Now, using equations (3) and (4) in equation (2), we have

$$[f(x)]^3 = f(x^3) + f(x).$$

(*iii*) We have, $\quad f(x) = \frac{16^x}{16^x+4}$ ...(1)

$$\therefore \quad f(1-x) = \frac{16^{1-x}}{16^{1-x}+4} = \frac{16\,.\,16^{-x}}{16\,.\,16^{-x}+4} = \frac{\frac{16}{16^x}}{\frac{16}{16^x}+4}$$

$$= \frac{16}{16+16^x(4)} = \frac{4}{4+16^x} \quad ...(2)$$

Adding equations (1) and (2), we have

$$f(x) + f(1-x) = \frac{16^x}{16^x + 4} + \frac{4}{4 + 16^x} = \frac{4 + 16^x}{4 + 16^x} = 1.$$

NOTES

**Example 3.** (*i*) *If* $f(x) = \frac{1+x}{1-x}$*, show that* $f(f(\tan\theta)) = -\cot\theta$.

(*ii*) *If* $f(x) = \frac{1+e^x}{1-e^x}$*, show that* $f(-x) = -f(x)$.

**Solution.** (*i*) We have, $f(x) = \frac{1+x}{1-x}$

$$\therefore \quad f(\tan\theta) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\Rightarrow \quad f(f(\tan\theta)) = \frac{1+f(\tan\theta)}{1-f(\tan\theta)}$$

$$= \frac{1+\dfrac{1+\tan\theta}{1-\tan\theta}}{1-\dfrac{1+\tan\theta}{1-\tan\theta}} = \frac{1-\tan\theta+1+\tan\theta}{1-\tan\theta-1-\tan\theta}$$

$$= \frac{2}{-2\tan\theta} = -\frac{1}{\tan\theta} = -\cot\theta.$$

(*ii*) We have, $f(x) = \frac{1+e^x}{1-e^x}$

$$\therefore \quad f(-x) = \frac{1+e^{-x}}{1-e^{-x}} = \frac{1+\dfrac{1}{e^x}}{1-\dfrac{1}{e^x}} = \frac{e^x+1}{e^x-1} = -\left(\frac{1+e^x}{1-e^x}\right) = -f(x).$$

**Example 4.** *Find the domain of the following functions :*

(*i*) $f(x) = \frac{1}{\sqrt{x+|x|}}$ (*ii*) $f(x) = \sin^{-1} 2x$

(*iii*) $f(x) = \cos^{-1}(3x-1)$ (*iv*) $f(x) = \tan^{-1}(2x+1)$

(*v*) $f(x) = \frac{1}{\sqrt{x+[x]}}$ (*vi*) $f(x) = |x-2|$.

**Solution.** (*i*) We have, $f(x) = \frac{1}{\sqrt{x+|x|}}$

The function $f(x)$ is defined only when $x + |x|$ is positive.
We know that,

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x + |x| = \begin{cases} x+x, & \text{if } x \geq 0 \\ x-x, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x + |x| = \begin{cases} 2x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

NOTES

$\therefore$ $x + |x|$ is positive only when $|x| = x$ *i.e.,* when $x > 0$.

$\therefore$ Domain of $f = D(f)$ = the set of all positive reals *i.e.,* $(0, \infty)$.

(*ii*) We have, $f(x) = \sin^{-1} 2x$.

Since, $\sin^{-1} x$ is defined only for $x \in [-1, 1]$.

$\therefore$ $f(x) = \sin^{-1} 2x$ is defined only if :

$$-1 \le 2x \le 1$$

$$\Rightarrow \quad -\frac{1}{2} \le x \le \frac{1}{2}$$

$$\therefore \quad \text{Domain of } f = D(f) = \left[-\frac{1}{2}, \frac{1}{2}\right].$$

(*iii*) We have, $f(x) = \cos^{-1}(3x - 1)$.

Since, the domain of $\cos^{-1} x$ is $[-1, 1]$.

$\therefore$ $f(x) = \cos^{-1}(3x - 1)$ is defined only if

$$-1 \le 3x - 1 \le 1$$

$$\Rightarrow \quad 0 \le 3x \le 2$$

$$\Rightarrow \quad 0 \le x \le \frac{2}{3}$$

$$\therefore \quad \text{Domain of } f = D(f) = \left[0, \frac{2}{3}\right].$$

(*iv*) We have, $f(x) = \tan^{-1}(2x + 1)$

Since, the domain of $\tan^{-1} x$ is the set of all reals *i.e.,* $(-\infty, \infty)$

$\therefore$ $f(x) = \tan^{-1}(2x + 1)$ is defined only if :

$$-\infty < 2x + 1 < \infty$$

$$\Rightarrow \quad -\infty < x < \infty$$

$\therefore$ Domain of $f = D(f) = (-\infty, \infty)$. *i.e.,* the set of all reals.

(*v*) We have, $f(x) = \dfrac{1}{\sqrt{x + [x]}}$

Since
$$[x] = \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

$$\Rightarrow \quad x + [x] = \begin{cases} > 0 & \text{if } x > 0 \\ = 0 & \text{if } x = 0 \\ < 0 & \text{if } x < 0 \end{cases}$$

$\therefore$ $f(x)$ is defined for all values of $x$ for which $x + [x] > 0$

$\therefore$ Domain of $f = D(f) = (0, \infty)$.

(*vi*) We have, $f(x) = |x - 2|$.

Since, $f(x)$ is defined for all values of $x$ and has real, unique and finite values.

$\therefore$ Domain of $f = D(f)$ = the set of all reals. *i.e.,* $(-\infty, \infty)$.

**Example 13.** *Find the domain and range of the following functions :*

(*i*) $3 \sin x - 4 \cos x$ (*ii*) $\dfrac{1}{\sqrt{x+2}}$ (*iii*) $1 + x - [x - 2]$

(*iv*) $\dfrac{x^2 - 9}{x - 3}$ (*v*) $\dfrac{1}{2 - \cos 3x}$.

**Solution.** (*i*) Let $y = f(x) = 3 \sin x - 4 \cos x$ ...(1)

**Domain.** Since, $\sin x$ and $\cos x$ are defined for all real values of $x$.

$\therefore$ $3 \sin x - 4 \cos x$ is also defined for all real values of $x$.

$$\Rightarrow \quad D(f) = R.$$

**Range.** From equation (1), we have

$$y = 3 \sin x - 4 \cos x \quad ...(2)$$

Put $3 = r \cos \theta$ and $4 = r \sin \theta$, $r > 0$. Square and add,

$$(3)^2 + (4)^2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\Rightarrow \quad 25 = r^2 \quad [\because \sin^2 A + \cos^2 A = 1]$$

$$\Rightarrow \quad r = 5$$

$\therefore$ From equation (2), we have

$$y = r \cos \theta \sin x - r \sin \theta \cos x$$

$$\Rightarrow \quad y = 5 (\cos \theta \sin x - \sin \theta \cos x)$$

$$[\because \sin (A - B) = \sin A \cos B - \cos A \sin B]$$

$$\Rightarrow \quad y = 5 \sin (x - \theta).$$

But, $-1 \leq \sin (x - \theta) \leq 1$

$$\Rightarrow \quad -5 \leq 5 \sin (x - \theta) \leq 5$$

$$\Rightarrow \quad -5 \leq y \leq 5 \quad \Rightarrow \quad -5 \leq f(x) \leq 5$$

$$\therefore \quad R(f) = [-5, 5].$$

(*ii*) Let $y = f(x) = \dfrac{1}{\sqrt{x+2}}$ ...(1)

**Domain.** Here, $f(x)$ is defined only for those real values of $x$ for which $x + 2 > 0$. *i.e.,* $x > -2$.

$$\therefore \quad D(f) = [-2, \infty).$$

**Range.** From equation (1), we have

$$y = \frac{1}{\sqrt{x+2}}$$

Since, $\sqrt{x+2}$ is the positive square root of $x + 2$ for all $x$ in domain of $f(x)$.

$$\Rightarrow \quad \frac{1}{\sqrt{x+2}} > 0 \quad i.e., \; y > 0.$$

$$\therefore \quad R(f) = (0, \infty).$$

**Alternatively,**

From equation (1), we have

$$y = \frac{1}{\sqrt{x+2}} \quad \Rightarrow \quad y^2 = \frac{1}{x+2}$$

$$\Rightarrow \quad x + 2 = \frac{1}{y^2} \quad \Rightarrow \quad x = \frac{1}{y^2} - 2$$

But $x > -2$, $\Rightarrow \dfrac{1}{y^2} - 2 > -2$

$$\Rightarrow \quad \frac{1}{y^2} > 0 \quad \text{or} \quad y^2 > 0 \quad \Rightarrow \quad \text{either } y > 0 \quad \text{or} \quad y < 0$$

$$\Rightarrow \quad y > 0 \quad \therefore \quad R(f) = (0, \infty).$$

NOTES

(*iii*) Let $y = f(x) = 1 + x - [x - 2]$ ...(1)

**Domain.** Since, $f(x)$ is the difference of two functions.

$\therefore$ Let $g(x) = 1 + x$ and $h(x) = [x - 2]$

$\therefore$ $D(g) = R$ and $D(h) = R$

$\therefore$ $D(f) = D(g) \cap D(h) = R \cap R = R.$

**Range.** We know that

$$[a] \leq a < [a] + 1$$

$$\Rightarrow \quad 0 \leq a - [a] < 1$$

Now, put $a = x - 2$, we have

$$0 \leq x - 2 - [x - 2] < 1$$

$$\Rightarrow \quad 3 \leq 1 + x - [x - 2] < 4 \qquad \text{[Adding 3 throughout]}$$

$$\Rightarrow \quad 3 \leq f(x) < 4$$

$$\therefore \quad R(f) = [3, 4).$$

(*iv*) Let $y = f(x) = \dfrac{x^2 - 9}{x - 3}$ ...(1)

**Domain.** Clearly, $f(x)$ is defined for all those real values of $x$ for which $x - 3 \neq 0$, *i.e.,* $x \neq 3$.

$$\therefore \quad (f) = R - \{3\}.$$

**Range.** From equation (1), we have

$$y = \frac{x^2 - 9}{x - 3}$$

$$\Rightarrow \quad y = \frac{(x + 3)(x - 3)}{x - 3} \quad \Rightarrow \quad y = x + 3$$

$\therefore$ When $x = 3$, then, $y = 3 + 3 = 6$.

$$\therefore \quad R(f) = R - \{6\}.$$

(*v*) Let $y = f(x) = \dfrac{1}{2 - \cos 3x}$ ...(1)

**Domain.** The function $f(x)$ is not defined for those values of $x$ for which $2 - \cos 3x = 0$.

$$\therefore \quad 2 - \cos 3x \neq 0$$

which is true for all real values of $x$.

$$\because \quad -1 \leq \cos 3x \leq 1$$

$$\therefore \quad D(f) = R.$$

**Range.** From equation (1), we have

$$y = \frac{1}{2 - \cos 3x} \quad \Rightarrow \quad 2 - \cos 3x = \frac{1}{y}$$

$$\Rightarrow \quad \cos 3x = 2 - \frac{1}{y} \qquad [\because \ -1 \leq \cos 3x \leq 1]$$

$$\Rightarrow \quad -1 \leq 2 - \frac{1}{y} \leq 1$$

NOTES

$\Rightarrow \quad -3 \leq -\frac{1}{y} \leq -1$ [Adding – 2 throughout]

$\Rightarrow \quad 3 \geq \frac{1}{y} \geq 1 \;\Rightarrow\; \frac{1}{3} \leq y \leq 1$

$\Rightarrow \quad y \in \left[\frac{1}{3}, 1\right]$

$\therefore \quad R(f) = \left[\frac{1}{3}, 1\right].$

**Example 6.** *Find the domain and range of the following functions :*

(*i*) $f(x) = \begin{cases} x+7 & \text{if } -3 \leq x < 5 \\ x^2 & \text{if } 5 \leq x < 7 \\ 6-2x & \text{if } x \geq 7 \end{cases}$ (*ii*) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}.$

**Solution.** (*i*) We have, $f(x) = \begin{cases} x+7 & \text{if } -3 \leq x < 5 \\ x^2 & \text{if } 5 \leq x < 7 \\ 6-2x & \text{if } x \geq 7 \end{cases}$

**Domain.** Clearly, $f(x)$ is defined only when ;

Either $-3 \leq x < 5$ or $5 \leq x < 7$ or $x \geq 7$

*i.e.,* $x \in [-3, 5)$ or $x \in [5, 7)$ or $x \in [7, \infty)$.

$\therefore \quad D(f) = [-3, 5) \cup [5, 7) \cup [7, \infty) = [-3, \infty)$

**Range.** When $-3 \leq x < 5$, $f(x) = x + 7$

$\Rightarrow \quad -3 + 7 \leq x + 7 \leq 5 + 7$

$\Rightarrow \quad 4 \leq f(x) \leq 12$

$\therefore \quad f(x) \in [4, 12]$

When $5 \leq x < 7$, $\quad f(x) = x^2$

$\Rightarrow \quad (5)^2 \leq x^2 < (7)^2$

$\Rightarrow \quad 25 \leq x^2 < 49$

$\Rightarrow \quad f(x) \in [25, 49).$

When $x \geq 7$, $\quad f(x) = 6 - 2x$

$\Rightarrow \quad -2x \leq -2\,(7)$

$\Rightarrow \quad 6 - 2x \leq 6 - 14$

$\Rightarrow \quad f(x) \leq -8$

$\Rightarrow \quad f(x) \in (-\infty, -8]$

$\therefore \quad R(f) = [4, 12] \cup [25, 49) \cup (-\infty, -8]$

$= (-\infty, -8] \cup [4, 12] \cup [25, 49).$

(*ii*) We have, $\quad f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}.$

**Domain.** Since, the given function $f(x)$ is defined for all real values of $x$

$\therefore \quad D(f) = R.$ *i.e.,* the set of all reals.

NOTES

**Range.** When $x < 0$ ; $\therefore$ $f(x) = x^2$

$\Rightarrow$ $x^2 > 0$

$\Rightarrow$ $f(x) \in (0, \infty)$.

When $0 \le x \le 1$ ; $f(x) = x$

$\Rightarrow$ $0 \le x \le 1$

$\Rightarrow$ $f(x) \in [0, 1]$

When $x > 1$ ; $f(x) = \dfrac{1}{x}$

$\Rightarrow$ $\dfrac{1}{x} < 1$ and $\dfrac{1}{x} > 0$

$\Rightarrow$ $f(x) \in (0, 1)$.

$\therefore$ $R(f) = (0, \infty) \cup [0, 1] \cup (0, 1) = [0, \infty)$.

**Example 7.** (*i*) *If* $f(x) = \cos x$ *and* $g(x) = e^x$, *find* $(f + g)$, $(f - g)$, $(f \,.\, g)$ *and* $\left(\dfrac{f}{g}\right)$.

(*ii*) *If* $f(x) = e^x$ *and* $g(x) = \log_e x$, *find fog and gof.*

**Solution.** (*i*) We have,

$$f(x) = \cos x \quad \text{and} \quad g(x) = e^x$$

$\therefore$
$$(f + g)(x) = f(x) + g(x) = \cos x + e^x$$
$$(f - g)(x) = f(x) - g(x) = \cos x - e^x$$
$$(f \,.\, g)(x) = f(x) \,.\, g(x) = \cos x \,.\, e^x = e^x \cos x$$
$$\left(\frac{f}{g}\right) x = \frac{f(x)}{g(x)} = \frac{\cos x}{e^x} = e^{-x} \cos x$$

(*ii*) We have, $f(x) = e^x$ and $g(x) = \log_e x$

Since, $D(f)$ = the set of all reals *i.e.,* R

$R(f) = (0, \infty)$.

and $g(x) = \log_e (x)$ ; $x > 0$

$\therefore$ $D(g) = (0, \infty)$

and $R(g)$ = the set of all reals *i.e.,* R

$\therefore$ $R(g) = (0, \infty) \subseteq D(f)$

$\therefore$ $(fog)(x) = f(g(x)) = f(\log_e x) = e^{\log_e x} = x$

$[\because \; e^{\log_e f(x)} = f(x)]$

Also, $R(f) = (0, \infty) = D(g)$

$\therefore$ $(gof)(x) = g(f(x)) = g(e^x)$

$= \log_e . e^x = x \log_e . e = x$ $[\because \; \log_e . e = 1]$

**Example 8.** *Let f be an exponential function and g be the logarithmic function. Then find :*

(*a*) $(f + g)(x)$ (*b*) $(f - g)(x)$ (*c*) $(f.g)(x)$

(*d*) $\left(\dfrac{f}{g}\right)(x)$ (*e*) $(fog)(x)$ (*f*) $(gof)(x)$

(*g*) $(fof)(x)$ (*h*) $(f + g)(1)$.

**Solution.** We have, $f(x) = e^x$ and $g(x) = \log x$

(*a*) $\therefore$ $(f + g)(x) = f(x) + g(x) = e^x + \log x$

(*b*) $(f - g)(x) = f(x) - g(x) = e^x - \log (x)$

(*c*) $(f \,.\, g)(x) = f(x) \,.\, g(x) = e^x \,.\, \log x$

(*d*) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{e^x}{\log x}$

(*e*) $(fog)(x)$

As $f(x) = e^x$

$\therefore$ D($f$) = R and R($f$) = set of all positive reals = $R^+$

$g(x) = \log x$

$\therefore$ D($g$) = set of all positive reals *i.e.,* $R^+$ and R($g$) = R

Since, $R(g) = R \subseteq D(f)$

$\therefore$ $(fog)(x) = f(g(x))$

$= f(\log x) = e^{\log x} = x$

(*f*) $(gof)(x)$

Since, $R(f) = R^+ \subseteq D(g)$

$\therefore$ $(gof)(x) = g(f(x)) = g(e^x) = \log e^x = x \log e = x$

(*g*) $(fof)(x)$

Since, $R(f) = R^+ \subseteq D(f)$

$\therefore$ $(fof)(x) = f(f(x)) = f(e^x) = e^{(e^x)}$

(*h*) $(f + g)(x) = f(x) + g(x) = e^x + \log x$

$\Rightarrow$ $(f + g)(1) = e^1 + \log 1 = e$ $[\because \log 1 = 0]$

**Example 9.** *Find fog and gof, if*

(*i*) $f(x) = \tan x,\ x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ *and* $g(x) = \sqrt{1 - x^2}$

(*ii*) $f(x) = [x]$ *and* $g(x) = \sin (\pi x)$.

**Solution.** (*i*) We have,

$$f(x) = \tan x \ ; \quad x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \text{ and } g(x) = \sqrt{1 - x^2}$$

$\therefore$ $(fog)(x) = f(g(x)) = f(\sqrt{1 - x^2})$, if $-1 < x \le 1$

$= \tan \sqrt{1 - x^2}$.

and $(gof)(x) = g(f(x)) = g(\tan x)$

$= \sqrt{1 - \tan^2 x}$, $\dfrac{-\pi}{4} < x \le \dfrac{\pi}{4}$.

(*ii*) We have, $f(x) = [x]$ and $g(x) = \sin (\pi x)$

$\therefore$ $(fog)(x) = f(g(x)) = f(\sin (\pi x)) = [\sin (\pi x)]$

and $(gof)(x) = g(f(x))$

$= g([x]) = \sin (\pi [x])$

$[\because \sin (n\pi) = 0$, if $n$ is an integer$]$

$= 0$ for all values of $x$.

**NOTES**

**Example 10.** *If* $f(x) = \frac{x-1}{x+1}$, *$(x \neq 1, -1)$, show that $fof^{-1}$ is an identity function.*

**Solution.** We have, $f(x) = \frac{x-1}{x+1}$, $(x \neq +1, -1)$

**f(x) is one-one :**

Let $f(x_1) = \frac{x_1 - 1}{x_1 + 1}$ and $f(x_2) = \frac{x_2 - 1}{x_2 + 1}$

$f(x_1) = f(x_2)$

$\Rightarrow \quad \frac{x_1 - 1}{x_1 + 1} = \frac{x_2 - 1}{x_2 + 1}$

$\Rightarrow \quad (x_1 - 1)(x_2 + 1) = (x_1 + 1)(x_2 - 1)$ [cross-multiplication]

$\Rightarrow \quad x_1x_2 + x_1 - x_2 - 1 = x_2x_1 + x_2 - x_1 - 1$

$\Rightarrow \quad x_1 = x_2$

$\therefore$ $f(x)$ is one-one.

**f(x) is onto :**

Let $f(x) = y$

$\Rightarrow \quad \frac{x-1}{x+1} = y \quad \Rightarrow \quad (x-1) = y(x+1)$

$\Rightarrow \quad x - 1 = xy + y \quad \Rightarrow \quad x - xy = 1 + y$

$\Rightarrow \quad x(1-y) = 1 + y$

$\Rightarrow \quad x = \frac{1+y}{1-y}$

such that ; $f(x) = y$

$$\Rightarrow \quad f\left(\frac{1+y}{1-y}\right) = \frac{\frac{1+y}{1-y} - 1}{\frac{1+y}{1-y} + 1} = \frac{(1+y)-(1-y)}{(1+y)+(1-y)} = \frac{2y}{2} = y$$

$\therefore$ $f$ is onto.

$\therefore$ $f(x)$ is one-one and onto, so $f(x)$ is invertible and hence $f^{-1}$ exists.

Since, $f^{-1}$ exists.

$\therefore \quad y = f(x)$

$\Rightarrow \quad x = f^{-1}(y)$

$\therefore$ We have, $x = \frac{1+y}{1-y}$

$\Rightarrow \quad f^{-1}(y) = \frac{1+y}{1-y}$

$\Rightarrow \quad f^{-1}(x) = \frac{1+x}{1-x}$ for all $x \in \mathrm{R} - \{1\}$.

Now, we have to show that $fof^{-1}$ is an identity function.

$\therefore \quad (fof^{-1})(x) = f(f^{-1}(x))$

$$= f\left(\frac{1+x}{1-x}\right) = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} = \frac{(1+x)-(1-x)}{(1+x)+(1-x)} = \frac{2x}{2}$$

$$= x \text{ for all } x \in R - \{1\}.$$

$\therefore$ $fof^{-1}$ is an identity function.

**Example 11.** *Determine whether the following functions are even or odd.*

(*i*) $\cos x + 4 \sec x + 3x^4$ (*ii*) $e^x - e^{-x} + \sin x$

(*iii*) $\sin x + \cos x$ (*iv*) $x^2 - |x|$.

**Solution.** (*i*) Let $f(x) = \cos x + 4 \sec x + 3x^4$

$\therefore \quad f(-x) = \cos(-x) + 4 \sec(-x) + 3(-x)^4$

$[\because \cos(-\theta) = \cos\theta, \sec(-\theta) = \sec\theta]$

$= \cos x + 4 \sec x + 3x^4 = f(x)$

$\Rightarrow \quad f(-x) = f(x)$

$\therefore$ $f(x)$ is an even function.

(*ii*) Let $f(x) = e^x - e^{-x} + \sin x$

$\therefore \quad f(-x) = e^{-x} - e^{-(-x)} + \sin(-x) \quad [\because \sin(-\theta) = -\sin\theta]$

$= e^{-x} - e^{x} - \sin x = -[e^x - e^{-x} + \sin x] = -f(x)$

$\therefore \quad f(-x) = -f(x)$

$\therefore$ $f(x)$ is an odd function.

(*iii*) Let $f(x) = \sin x + \cos x$

$\therefore \quad f(-x) = \sin(-x) + \cos(-x)$

$[\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta]$

$= -\sin x + \cos x = \cos x - \sin x$

$\therefore$ $(\cos x - \sin x)$ is neither equal to $f(x)$ nor equal to $-f(x)$.

So, $f(x)$ is neither even nor odd.

(*iv*) Let $f(x) = x^2 - |x|$

$\therefore \quad f(-x) = (-x)^2 - |-x| = x^2 - |x| = f(x)$

$\therefore \quad f(-x) = f(x)$

$\therefore$ $f(x)$ is an even function.

## EXERCISE

**1.** Let $f$ and $g$ be real functions defined by

$f(x) = \sqrt{1+x}$ and $g(x) = \sqrt{1-x}$. Then, find each of the following functions :

(*i*) $f + g$ (*ii*) $f - g$ (*iii*) $f \cdot g$ (*iv*) $\frac{f}{g}$.

NOTES

2. Let $f$ and $g$ be real functions defined by $f(x) = \frac{1}{x+4}$ and $g(x) = (x+4)^3$. Then, find each of the following functions :

(i) $f + g$  (ii) $f - g$  (iii) $f \cdot g$  (iv) $\frac{f}{g}$.

3. Find the domain of the following functions :

(i) $|x - 2|$  (ii) $e^{x+2}$  (iii) $\sin^{-1}(3x - 1)$

(iv) $e^{x + \sin x}$  (v) $\sqrt{x^2 - 1} + \frac{1}{\sqrt{x}}$  (vi) $x - [x]$.

4. Find the domain and range of $\frac{1}{2 - \cos 3x}$.

5. Find the domain and range of the following functions :

(i) $\frac{x^2}{2 + x^2}$  (ii) $\frac{x-1}{x+1}$  (iii) $[2 \cos x]$

(iv) $\frac{|x|}{x}$  (v) $\frac{x-2}{|x-2|}$  (vi) $\frac{1}{\sqrt{x+2}}$

(vii) $3 \sin x + 4 \cos x + 1$  (viii) $\frac{|x-3|}{x-3}$.

6. If the map $f : R \to R$ is given by $f(x) = \log(1 + x)$ and the map $g : R \to R$ is given by $g(x) = e^x$.
Find $(gof)(x)$ and $(fog)(x)$.

7. If $f(x) = \sqrt{1-x}$ and $g(x) = \log_e x$ are two real functions, then describe functions $fog$ and $gof$.

8. If $f(x) = [x]$ and $g(x) = |x|$.
Find (i) $fog(x)$  (ii) $gof(x)$  (iii) $gog(x)$.

9. If $f(x) = \frac{x-1}{x+1}$, $(x \neq 1, -1)$, show that $fof^{-1}$ is an identity function.

10. If $f(x) = \frac{1}{1-x}$, show that $(fof)\left(\frac{1}{2}\right) = -1$.

11. Show that $f(x) = \tan^2 x + |x|$ is an even function.

12. Find the inverse of the function $f(x) = 4x - 7$, $x \in R$.

13. Find the period of the function $f(x) = \sin 3x + \cos 4x$.

## Answers

1. (i) $\sqrt{1+x} + \sqrt{1-x}$, $x \in [-1, 1]$  (ii) $\sqrt{1+x} - \sqrt{1-x}$, $x \in [-1, 1]$

(iii) $\sqrt{1-x^2}$, $x \in [-1, 1]$  (iv) $\sqrt{\frac{1+x}{1-x}}$, $x \in [-1, 1]$

2. (i) $\frac{(x+4)^4 + 1}{x+4}$  (ii) $\frac{1-(x+4)^4}{x+4}$  (iii) $(x+4)^2$

(iv) $\frac{1}{(x+4)^4}$

3. (i) R  (ii) R  (iii) $\left[0, \frac{2}{3}\right]$

(iv) R  (v) $[1, \infty)$  (vi) R

NOTES

4. **Domain.** R

**Range.** $\left[\frac{1}{3}, 1\right]$

5. (*i*) **Domain.** R

**Range.** [0, 1)

(*ii*) **Domain.** R – {– 1}

**Range.** R – {1}

(*iii*) **Domain.** R

**Range.** {– 2, – 1, 0, 1, 2}

(*iv*) **Domain.** R – {0}

**Range.** {1, – 1}.

(*v*) **Domain.** R – {2}

**Range.** {– 1, 1}

(*vi*) **Domain.** (– 2, ∞)

**Range.** (0, ∞)

(*vii*) **Domain.** R

**Range.** [– 4, 6]

(*viii*) **Domain.** R – {3}

**Range.** {1, – 1}.

6. $1 + x$ ; $\log (e^x + 1)$

7. $\sqrt{1 - \log_e x}$ ; $\frac{1}{2} \log (1 - x)$

8. (*i*) $[\,|x|\,]$ (*ii*) $|\,[x]\,|$ (*iii*) $|x|$

12. $f^{-1}(x) = \frac{x + 7}{4}, x \in$ R.

13. $2\pi$

NOTES

# LIMIT OF FUNCTIONS

**STRUCTURE**



## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Preliminaries and Results on Limit
- Algebraic Operations of Functions
- Algebra of Limits of Functions: Theorem
- Some Useful Limits

## INTRODUCTION

The most important idea in Calculus is that of limit. The limit concept is at the foundation of almost all of mathematical analysis and an understanding of it is absolutely essential. In this chapter, we give some important results on limits. We shall deal with properties of continuous functions and uniform continuity.

## PRELIMINARIES AND RESULTS ON LIMIT

(*i*) Let X by any set.

*A function* $f : X \to R$, *the set of real numbers, is called a* **real valued** *function.*

*A function* $f : A\ (\subseteq R) \to X$ *is called a function of a* **real variable.**

*A function from a subset of R into R, is called a* **real valued function** *of a* **real variable.**

In this chapter, we shall deal with only real valued functions of a real variable.

(*ii*) Let $x$ be a real variable and $a$ be a fixed finite number. Let $x$ pass through an infinite number of values according to some rule. If the successive values approach $a$

in such a way that $| x - a |$ becomes and remains smaller than any preassigned number $\varepsilon > 0$ (however small), then we say that '*x tends to a*' or '*x has the limit a*' and we write $x \to a$ or $\lim x = a$.

The definition implies that for each $\varepsilon > 0$, there exists a stage in the successive values of $x$ such that all the values of $x$ after this stage lie in the interval $(a - \varepsilon, a + \varepsilon)$. The definition does not imply that $x$ must take the value $a$.

(*iii*) If $x$ approaches to $a$ taking all values less than $a$, then we say that *x tends to a from the left* and write $x \to a^-$.

(*iv*) If $x$ approaches to $a$, taking all values greater than $a$, then we say that $x$ tends to $a$ from the right and write $x \to a^+$.

(*v*) If a real variable $x$ takes successively values which ultimately become and remain greater than every positive number (however large), then *we say that x tends to plus infinity* and we write $x \to + \infty$ or just $x \to \infty$. That is, given any positive number $k$ (however large), there exists a stage in the succession of values of $x$ after which all values of $x$ are larger than $x$.

(*vi*) If the successive values of $x$ become and remain smaller than every negative number (however small), then *we say that x tends to minus infinity and write* $x \to \infty$.

## Limit of a Function

Let $f$ be a function defined in some interval containing the point $a$, but may or may not be defined at $a$ itself. We consider the behaviour of $f(x)$ as $x \to a$. It may happen that the values of $f$ become closer and closer to a number $l$ as $x \to a$, *i.e.*, the absolute value of the difference $f(x) - l$ can be made smaller than any pre-assigned positive number $\varepsilon$, however small, by taking $x$ sufficiently close to $a$. In such a case, *we say that f(x) approaches or converges or tends to the limit l as* $x \to a$ and we write

$\lim_{x \to a} f(x) = l$ or $f(x) \to l$ as $x \to a$. Formally, we define :

**Definition.** *Let f be a function defined in a neighbourhood of a except possible at a. Then, a real number l is said to be a limit of f as x tends to a if given any* $\varepsilon > 0$, *however small, there exists* $\delta > 0$ *(depending upon* $\varepsilon$*) such that*

$$| f(x) - l | < \varepsilon, \textit{ whenever } 0 < | x - a | < \delta.$$

*i.e.*,
$$l - \varepsilon < f(x) < l + \varepsilon, \text{ whenever } x \in (a - \delta, a) \cup (a, a + \delta)$$

We write $\lim_{x \to a} f(x) = l$ or $f(x) - l$ as $x \to a$.

**Remarks 1.** The limit of $f$ at $a$, if exists, will continue to exist and be the same if we change the value of $f$ at $a$ only.

**2.** In order to show that $\lim_{x \to a} f(x) \neq l$, it is enough to produce one $\varepsilon > 0$, such that for each $\delta > 0$, there is some $x$ for which

$$0 < | x - a | < \delta \text{ and } | f(x) - l | \geq \varepsilon.$$

## Right Hand and Left Hand Limits

The above definition of limit implies that $f(x)$ approaches the same limit $l$ irrespective of the manner in which $x$ approaches $a$ whether from right or from left. However, $f(x)$ may tend to the limit $l$ as $x \to a$ from the right only, *i.e.*, $x \to a^+$, then we call this the right hand limit of $f$ at $x = a$, *i.e, a real number l is said to be the right* *hand limit of f as* $x \to a^+$ *if* given any $\varepsilon > 0$, however small, there exists a $\delta > 0$ (depending upon $\varepsilon$) such that

$| f(x) - l | < \varepsilon$ whenever $a < x < a + \delta$ *i.e.*, $l - \varepsilon < f(x) < l + \varepsilon$ whenever $a < x < a + \delta$. We write $\lim_{x \to a^+} f(x) = l$ or $f(a^+) = l$. Likewise, a real number $l$ is said to be the left hand limit of $f$ as $x \to a^-$ if given any $\varepsilon > 0$, however small ; there exists a $\delta > 0$ (depending upon $\varepsilon$) such that

NOTES

$$| f(x) - l | < \varepsilon \qquad \text{whenever } a - \delta < x < a$$

*i.e.*,
$$l - \varepsilon < f(x) < l + \varepsilon \qquad \text{whenever } a - \delta < x < a.$$

We write $\lim_{x \to a^-} f(x) = l$ or $f(a^-) = l$

Clearly, if $\lim_{x \to a} f(x) = l$, then both the left hand and right hand limits exist and each is equal to $l$. Conversely, if both the right hand and left hand limits exist and are equal, then it can be easily seen that $\lim_{x \to a} f(x)$ exists and is equal to them. Hence, a necessary and sufficient condition for limit of a function to exist is that

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

## Theorem

*Limit of a function at a point, if exists, is unique.*

**Proof.** Let $\lim_{x \to a} f(x)$ exist.

Let, if possible, $f(x)$ tend to two different limits $l$ and $l'$ as $x \to a$.

Take $\varepsilon = \frac{1}{2} | l - l' | > 0.$

Since $f(x) \to l$ as $x \to a$, there exists $\delta_1 > 0$ such that

$$| f(x) - l | < \varepsilon \text{ whenever } 0 < | x - a | < \delta_1$$

Since $f(x) \to l'$ as $x \to a$ there exists $\delta_2 > 0$ such that

$$| f(x) - l' | < \varepsilon \text{ whenever } 0 < | x - a | < \delta_2$$

Let $\delta = \min. \{\delta_1, \delta_2\}$. Then

$$\begin{aligned} | l - l' | &= | l - f(x) + f(x) - l' | \\ &\le | f(x) - l | + | f(x) - l' | \\ &< \varepsilon + \varepsilon \text{ whenever } 0 < | x - a | < \delta \\ &= | l - l' | \end{aligned}$$

This is a contradiction.

Hence, $l = l'$.

## Infinite Limits

Consider the behaviour of a function $f$ which increases continuously as $x \to a$. If the values of $f$ become and remain greater than any positive number, however large, for all $x$ sufficiently close to $a$, we say that $f(x)$ increases beyond limit or that it tends to $\infty$. If as $x \to a$, $f(x)$ decreases continuously beyond any limit, we say that $f(x)$ tends to $-\infty$. Formally we define :

**Definition.** *A function f is said to tend to ∞ (or diverge to ∞) if given any positive number k however large, there exists a $\delta > 0$ such that $f(x) > k$ whenever $0 < | x - a | < \delta$.*

We write $f(x) \to \infty$ as $x \to a$ or $\lim_{x \to a} f(x) = \infty$.

A function $f$ is said to tend to $-\infty$ (or diverges to $-\infty$) if given any positive number $k$, however large, there exists a $\delta > 0$ such that $f(x) < -k$ whenever $0 < | x - a | < \delta$.

We write $f(x) \to -\infty$ as $x \to a$ or $\lim_{x \to a} f(x) = -\infty$.

If a function $f$ does not tend to a finite limit or to ∞ or –∞, then (*i*) if it is bounded in a neighbourhood of $a$, it is said to *oscillate finitely*, (*ii*) if it is unbounded in a neighbourhood of $a$, it is said to *oscillate infinitely*.

**Examples.** (*i*) *Let* $f(x) = \frac{1}{x}$, $x \neq 0$.

Here $f(x)$ is not defined at $x = 0$.

Let $x \to 0^+$. As $x$ becomes smaller and smaller, $\frac{1}{x}$ becomes larger and larger and crosses all bounds. In fact, if $k$ is any positive number, however large, $\frac{1}{x} > k$ for $x < \frac{1}{k}$ $(= \delta)$.

Hence, whatever $k$ is given, we can find a $\delta > 0$ such that

$$f(x) = \frac{1}{x} > k \text{ for } 0 < x < \delta \left(= \frac{1}{k}\right).$$

Hence $\lim_{x \to 0^+} f(x) = \infty$

Similarly, $\lim_{x \to 0^-} f(x) = -\infty$

Since right hand limit is not equal to left hand limit, it follows that $\frac{1}{x}$ does not tend to any limit finite or infinite. As $\frac{1}{x}$ does not tend to any limit and is also not bounded in any neighbourhood of 0, therefore, $\frac{1}{x}$ oscillates infinitely as $x \to 0$.

(*ii*) Let $f(x) = \frac{1}{x^2}$, $x \neq 0$.

Here $\lim_{x \to 0^+} f(x) = \infty = \lim_{x \to 0^-} f(x)$.

## Limit of a Function as $x \to \infty$ or $x \to -\infty$

*A function f is said to tend to the limit l as $x \to \infty$ if given $\varepsilon > 0$, however small, there exists a positive number k (depending upon ε) such that*

$$| f(x) - l | < \varepsilon \quad \forall\, x \geq k$$

We write $\lim_{x \to +\infty} f(x) = l$ or $f(x) \to l$ as $x \to \infty$.



*A function f is said to tend to* $+\infty$ *as x tends to* $\infty$ *if given any positive number k, however large, there exists a positive number k′ such that* $f(x) > k \quad \forall\, x > k'$.

We write $\lim_{x \to \infty} f(x) = \infty$ or $f(x) \to \infty$ as $x \to \infty$

In a similar way, we can define

$$\lim_{x \to -\infty} f(x) = l \quad \text{and} \quad \lim_{x \to -\infty} f(x) = \infty, \text{ etc.}$$

**Examples.** (*i*) Let $f(x) = \frac{1}{x}$, $x \neq 0$. Then $f(x) \to 0$ as $x \to \infty$

(*ii*) Let $f(x) = \frac{1}{x^2}$, $x \neq 0$.

Then $f(x) \to 0$ as $x \to +\infty$,

$f(x) \to 0$ as $x \to -\infty$

and $f(x) \to +\infty$ as $x \to 0$.

(*iii*) Let $f(x) = -\frac{1}{x^2}$, $x \neq 0$

Then $f(x) \to -\infty$ as $x \to 0$.

## Algebraic Operations of Functions

Let $f : A \to R$ and $g : B \to R$ be two functions. Let $A \cap B \neq \phi$. We define on $A \cap B$ (*i.e.,* on the common portion of the domain of the functions $f$ and $g$), a new function.

(*i*) $f + g$ called the *sum of the functions f and g* by

$$(f + g)(x) = f(x) + g(x),\ x \in A \cap B.$$

(*ii*) $f - g$ called the difference of two functions $f$ and $g$ by

$$(f - g)(x) = f(x) - g(x),\ x \in A \cap B.$$

(*iii*) *fg called the product of two functions f and g* by

$$(fg)(x) = f(x) \,.\, g(x),\ x \in A \cap B.$$

(*iv*) $\frac{f}{g}$ *called the quotient of the functions f and g* by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ x \in A \cap B \text{ provided } g(x) \neq 0 \text{ for } x \in A \cap B.$$

If $c$ is a real number, then *the scalar product of f* by $c$ is the function $cf$ defined by

$$(cf)(x) = c \,.\, f(x), \quad \forall\, x \in A.$$

If $g(x) \neq 0 \quad \forall\, x \in B$, *then the reciprocal of g is the function* $\frac{1}{g}$ defined on B by

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)}.$$

**Remark.** Product of two functions is different from the composite of two functions. For example, if $f(x) = \sin x$ and $g(x) = x^2$ then $(fog)(x) = f(g(x)) = \sin x^2$ whereas $(fg)(x) = f(x)\, g(x) = x^2 \sin x$.

We shall now study the relation between the limits of functions and the limits of their sum, product, etc.



## Algebra of Limits of Functions : Theorem

*Let f and g be two functions defined in some deleted neighbourhood of a such that* $\lim\limits_{x \to a} f(x) = l$ *and* $\lim\limits_{x \to a} g(x) = m$.

Then,

(*i*) $\lim\limits_{x \to a} (f+g)(x) = l + m$ (*ii*) $\lim\limits_{x \to a} (f-g)(x) = l - m$

(*iii*) $\lim\limits_{x \to a} (fg)(x) = l \,.\, m$ (*iv*) $\lim\limits_{x \to a} (cf)(x) = cl$

(*v*) $\lim\limits_{x \to a} \left(\frac{1}{g}\right)(x) = \frac{1}{m}$ if $g(x) \neq 0,\ m \neq 0$

(*vi*) $\lim\limits_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$ > if $g(x) \neq 0,\ m \neq 0$.

**Proof.** (*i*) Let $\varepsilon > 0$ be given.

Now
$$\begin{aligned} |(f+g)(x) - (l+m)| &= |f(x) + g(x) - (l+m)| \\ &= |(f(x) - l) + g(x) - m| \\ &\le |f(x) - l| + |g(x) - m| \quad \ldots(1) \end{aligned}$$

We have to show that $|f(x) - l| + |g(x) - l| < \varepsilon$ for some $\delta$.

$\lim\limits_{x \to a} f(x) = l \Rightarrow \exists\, \delta_1 > 0$ such that

$$|f(x) - l| < \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_1 \quad \ldots(2)$$

$\lim\limits_{x \to a} g(x) = m \Rightarrow \exists\, \delta_2 > 0$ such that

$$|g(x) - m| < \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_2 \quad \ldots(3)$$

Let $\delta = \min. \{\delta_1, \delta_2\} > 0$.

Using (2) and (3) in (1), we have

$$|(f+g)(x) - (l+m)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

$$\Rightarrow \quad \lim_{x \to a} (f+g)(x) = l + m.$$

(*ii*) Proof is similar to (*i*).

(*iii*) Let $\varepsilon > 0$ be given.

Now
$$\begin{aligned} |(fg)(x) - lm| &= |f(x)\, g(x) - lm| \\ &= |f(x)\, g(x) - lg(x) + lg(x) - lm| \\ &= |g(x)\,(f(x) - l) + l(g(x) - m)| \\ &\le |g(x)|\, |f(x) - l| + |l|\, |g(x) - m| \quad \ldots(1) \end{aligned}$$

**NOTES**

$$\lim_{x \to a} g(x) = m \quad \Rightarrow \quad \exists\, \delta_1 > 0 \text{ such that}$$

$$| g(x) - m | < 1,\ 0 < | x - a | < \delta_1$$

$$\Rightarrow \quad | g(x) | = | g(x) - m + m |$$
$$\leq | g(x) - m | + | m |$$
$$< 1 + | m |,\ 0 < | x - a | < \delta_1 \qquad \text{...(2)}$$

$$\lim_{x \to a} f(x) = l \quad \Rightarrow \quad \exists\, \delta_2 > 0 \text{ such that}$$

$$| f(x) - l | < \frac{\varepsilon}{2(1+|m|)},\ 0 < | x - a | < \delta_2 \qquad \text{...(3)}$$

$$\lim_{x \to a} g(x) = m \quad \Rightarrow \quad \exists\, \delta_3 > 0 \text{ such that}$$

$$| g(x) - m | < \frac{\varepsilon}{2(|l|+1)},\ 0 < | x - a | < \delta_3 \qquad \text{...(4)}$$

Let $\delta = \min.\ \{\delta_1, \delta_2, \delta_3\}$. Using (2) to (4) in (1), we have

$$| (fg)(x) - lm | < \frac{\varepsilon}{2}\frac{(1+|m|)}{1+|m|} + |l|\frac{\varepsilon}{2(|l|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ 0 < | x - a | < \delta$$

$$\Rightarrow \quad \lim_{x \to a} (fg)(x) = lm.$$

(*iv*) Proof is simple and is left as an exercise for the reader.

(*v*) Let $\varepsilon > 0$ be given

Now $$\left| \frac{1}{g(x)} - \frac{1}{m} \right| = \frac{|g(x) - m|}{|g(x)|\,|m|} \qquad \text{...(1)}$$

We have to show that $\dfrac{|g(x) - m|}{|g(x)|\,|m|} < \varepsilon$ for some $\delta$.

Since $$\lim_{x \to a} g(x) = m,$$

We can find $\delta_1 > 0$ such that

$$| g(x) - m | < \frac{|m|}{2},\ 0 < | x - a | < \delta_1$$

Now $$| m | = | m - g(x) + g(x) |$$
$$\leq | m - g(x) | + | g(x) | < \frac{|m|}{2} + | g(x) |$$

$$\Rightarrow \quad | g(x) | > \tfrac{1}{2} | m |,\ 0 < | x - a | < \delta_1 \qquad \text{...(2)}$$

Again, $\lim_{x \to a} g(x) = m \quad \Rightarrow \quad \exists\, \delta_2 > 0$ such that

$$| g(x) - m | < \tfrac{1}{2} | m |^2 \varepsilon,\ 0 < | x - a | < \delta_2 \qquad \text{...(3)}$$

Let $\delta = \min.\ \{\delta_1, \delta_2\}$. Using (2) and (3) in (1), we have

$$\left| \frac{1}{g(x)} - \frac{1}{m} \right| < \frac{2|m|^2 \varepsilon}{2|m|.|m|} = \varepsilon,\ 0 < | x - a | < \delta$$

$$\Rightarrow \quad \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}.$$

(*vi*) By case (*v*) above,

$$\lim_{x \to a} g(x) = m,\ m \neq 0 \quad \Rightarrow \quad \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$$

Now $\qquad \lim_{x \to a} f(x) = l \quad \text{and} \quad \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$

$\therefore$ by case (*iii*),

$$\lim_{x \to a} \left\{ f(x) . \frac{1}{g(x)} \right\} = l \,.\, \frac{1}{m} = \frac{1}{m}.$$

$\Rightarrow \qquad \lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{1}{m}.$

**Remark.** $\lim_{x \to a} (f \pm g)(x)$, $\lim_{x \to a} (fg)(x)$ and $\lim_{x \to a} \left(\frac{f}{g}\right)(x)$ may exist even if neither $\lim_{x \to a} f(x)$ nor $\lim_{x \to a} g(x)$ exists.

For example, let *f* and *g* be defined as follows :

$$f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if } x < a \\ -1 & \text{if } x > a \end{cases}$$

Then $\qquad (f + g)(x) = 0 \ \forall \ x \neq a$

and $\qquad (fg)(x) = -1 = \left(\frac{f}{g}\right)(x) \ \forall \ x \neq a$

$$\lim_{x \to a} (f + g)(x) = 0, \ \lim_{x \to a} (fg)(x) = -1 = \lim_{x \to a} \left(\frac{f}{g}\right)(x)$$

But $\qquad \lim_{x \to a^-} f(x) = -1 \ \text{ and } \ \lim_{x \to a^+} f(x) = 1$

$\therefore \quad \lim_{x \to a} f(x)$ does not exist

Similarly, $\lim_{x \to a} g(x)$ does not exist.

Again, let *f* and *g* be defined as follows :

$$f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$$

$$g(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$$

Then $\qquad (f - g)(x) = 0 \quad \forall \ x \neq a$

$\Rightarrow \quad \lim_{x \to a} (f - g)(x) = 0$, but $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist.

**Theorem.** *If* $\lim_{x \to a} f(x) = l$, *then* $\lim_{x \to a} | f(x) | = | l |$.

**Proof.** Let $\varepsilon > 0$ be given

$$\lim_{x \to a} f(x) = l \quad \Rightarrow \quad \exists \ \delta > 0 \text{ such that}$$

$$| f(x) - l | < \varepsilon, \ 0 < | x - a | < \delta$$

Now $\qquad | \, | f(x) | - | l | \, | \leq | f(x) - l | < \varepsilon, \ 0 < | x - a | < \delta$

$$(\because \quad | a - b | \geq | \, | a | - | b | \, |)$$

NOTES

$$\Rightarrow \quad \lim_{x \to a} |f(x)| = |l|.$$

**Remark 1.** Converse of above theorem need not to true.

For example, let $f(x) = \begin{cases} -1 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$

Then $\quad |f(x)| = 1 \quad \forall\, x \neq a.$

$\lim_{x \to a} |f(x)| = 1$ but $\lim_{x \to a^-} f(x) = -1$ and $\lim_{x \to a^+} f(x) = 1$

$\therefore \lim_{x \to a} f(x)$ does not exist.

**2.** Converse of above theorem is true if $l = 0$.

**Theorem.** (*Squeeze principle*). *If $f(x) \leq h(x) \leq g(x)$ for $0 < |x - a| < \delta$ and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = l, \text{ then } \lim_{x \to a} h(x) = l.$$

**Proof.** Let $\varepsilon > 0$ be given. Then

$$\lim_{x \to a} f(x) = l \quad \Rightarrow \quad \exists\, \delta_1 > 0 \text{ such that}$$

$$|f(x) - l| < \varepsilon,\ 0 < |x - a| < \delta_1$$

*i.e.,* $\quad l - \varepsilon < f(x) < l + \varepsilon,\ 0 < |x - a| < \delta_1.$

$$\lim_{x \to a} g(x) = l \quad \Rightarrow \quad \exists\, \delta_2 > 0 \text{ such that}$$

$$|g(x) - l| < \varepsilon,\ 0 < |x - a| < \delta_2$$

*i.e.,* $\quad l - \varepsilon < g(x) < l + \varepsilon,\ 0 < |x - a| < \delta_2$

Let $\quad \delta_3 = \min \{\delta, \delta_1, \delta_2\}.$

Then $\quad l - \varepsilon < f(x) \leq h(x) \leq g(x) < l + \varepsilon,\ 0 < |x - a| < \delta_3$

*i.e.,* $\quad l - \varepsilon < h(x) < l + \varepsilon,\ 0 < |x - a| < \delta_3$

$$\Rightarrow \quad \lim_{x \to a} h(x) = l.$$

We state below some results on limits which can be easily proved.

(*i*) If $f(x) \geq 0\ \forall\, x$ in a deleted neighbourhood of $a$, and $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} f(x) \geq 0$.

(*ii*) If $f$ and $g$ are two functions defined in a deleted neighbourhood A of $a$ and $f(x) \geq g(x)\ \forall\, x \in$ A, then $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$.

(*iii*) If $\lim_{x \to a} f(x) = l > 0$, then

$$\lim_{x \to a} [f(x)]^{\lambda} = [\lim_{x \to a} f(x)]^{\lambda} = l^{\lambda},$$

$$\lim_{x \to a} b^{f(x)} = b^{\left(\lim_{x \to a} f(x)\right)} = b^l$$

and $$\lim_{x \to a} (\log f(x)) = \log (\lim_{x \to a} f(x)) = \log l.$$

NOTES

## SOME USEFUL LIMITS

(*i*) $\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e,$ (*ii*) $\lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^x = e$

(*iii*) $\lim_{x \to 0} (1+x)^{1/x} = e,$ (*iv*) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a \ (a > 0)$

(*v*) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (*vi*) $\lim_{x \to 0} \frac{\tan x}{x} = 1.$

### SOLVED EXAMPLES

**Example 1.** *Show by definition that* $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$

**Sol.** Here the function $f(x)$ is defined for $x \neq 2$.

Let $\varepsilon > 0$ be given

Now, $$| f(x) - 4 | = \left| \frac{x^2 - 4}{x - 2} - 4 \right|$$

$$= | x + 2 - 4 | = | x - 2 |$$

$\therefore$ we need to prove that

$$| x - 2 | < \varepsilon \quad \text{for } 0 < | x - 2 | < \delta$$

Take $\delta = \varepsilon$. Then $| f(x) - 4 | < \varepsilon$ whenever $0 < | x - 2 | < \delta$

Hence $$\lim_{x \to 2} f(x) = 4.$$

**Example 2.** *Show that the function* $f(x) = x \sin \frac{1}{x}$ *for* $x \neq 0$*, has limit 0 as* $x \to 0$.

**Sol.** Let $\varepsilon > 0$ be given

$$\left| x \sin \frac{1}{x} - 0 \right| = \left| x \sin \frac{1}{x} \right|$$

$$\leq | x | \qquad \left( \because \left| \sin \frac{1}{x} \right| \leq 1 \right)$$

$$< \varepsilon \text{ if } | x | < \varepsilon$$

$\therefore$ taking $\delta = \varepsilon$, we see that

$$\left| x \sin \frac{1}{x} - 0 \right| < \varepsilon, \text{ whenever } 0 < | x | < \delta$$

Hence $$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

**Example 3.** *Discuss the behaviour as* $x \to 2$*, of the function* $f(x)$*, where*

$$f(x) = \begin{cases} 2x + 1 & \text{when } x < 2 \\ 3x + 5 & \text{when } x > 2. \end{cases}$$

**Sol.** Here

$$f(2^-) = \lim_{x \to 2^-} (2x + 1) = 5$$

$$f(2^+) = \lim_{x \to 1^+} (3x + 5) = 11$$

**NOTES**

Since $f(2^-) \neq f(2^+)$, $\lim_{x \to 2} f(x)$ does not exist.

## EXERCISE

**1.** Verify the following limits :

(*i*) $\lim_{x \to 2} \dfrac{x^3 + 3}{2x^2 + 5} = \dfrac{11}{13}$ (*ii*) $\lim_{x \to 2} \dfrac{x^3 - 8}{x^2 - 4} = 3$

(*iii*) $\lim_{x \to \infty} \dfrac{x^3 + 5x^2 + 3x + 2}{2x^3 + 7x^2 + 4x - 3} = \dfrac{1}{2}$.

**2.** Evaluate the following limits :

(*i*) $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$ (*ii*) $\lim_{x \to 0} \dfrac{\sin 5x^\circ}{x}$

(*iii*) $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{2x}$.

**3.** Show that as $x \to 0$, $\left(\dfrac{2}{3}\right)^{\frac{1}{x^2}} \to 0$ but $\left(\dfrac{2}{3}\right)^{\frac{1}{x}}$ does not tend to any limit.

**4.** Find the limit of $\dfrac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_m}$ when $x \to \infty$ for (*i*) $m > n$, (*ii*) $m < n$ and (*iii*) $m = n$.

**5.** Justify or falsify the following statements :

(*i*) If limits of two function $f$ and $g$ do not exist as $x \to a$, then the limits of $f + g$ and $fg$ exist as $x \to a$.

(*ii*) If limits of $f + g$ and $f$ exist as $x \to a$, then the limit of $g$ exists as $x \to a$.

(*iii*) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} f(x) \cdot g(x)$ both exist, then $\lim_{x \to a} g(x)$ exists.

**6.** If $f(x) = g(x)$ for $0 < | x | < \delta$ for some $\delta > 0$, then prove that $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x)$. Is the converse true ?

**7.** Give examples of two functions $f$ and $g$ to show that $f(x) > g(x)$ but $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.

**8.** If $f(x) \leq g(x)$ and $\lim_{x \to a} g(x)$ exists, does it follow that $\lim_{x \to a} f(x)$ exist ?

**9.** If $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$

then show that $\lim_{x \to a} f(x)$ does not exist for any $a \in \mathbb{R}$.

# UNIT II



# CONTINUITY OF FUNCTIONS

NOTES

**STRUCTURE**



## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Discontinuity of a Function
- Bounds of a Function
- Theorems of Continuous Functions
- Monotonic Functions
- Uniform Continuity

## INTRODUCTION

While defining $\lim_{x \to a} f(x)$, the function $f$ may or may not be defined at $x = a$. Even if $f$ is defined at $x = a$, $\lim_{x \to a} f(x)$ may or may not be equal to the value of the function at $x = a$. If $\lim_{x \to a} f(x) = f(a)$, then we say that $f$ is continuous at $x = a$.

A formal definition is :

## DEFINITION

*Let $f$ be a function defined in a neighbourhood of $a$. Then $f$ is said to be continuous at $a$ if given any $\varepsilon > 0$, however small, there exists a $\delta > 0$ (depending upon $\varepsilon$) such that*

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

Since $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, therefore, a function $f$ is continuous at $x = a$ if and only if

$$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x).$$

This form of definition is often useful.

If $\lim_{x \to a^-} f(x) = f(a)$, then we say that $f$ *is continuous to the left of a (or left continuous at a).*

If $\lim_{x \to a^+} f(x) = f(a)$, then we say *that f is continuous to the right of a (or right continuous at a).*

A function $f$ is said to be *continuous in an open interval (a, b)* if it is continuous at every point of $(a, b)$.

A function *f is said to be continuous in a closed* interval $[a, b]$ if it is

(*i*) right continuous at $a$

(*ii*) continuous at every point of $(a, b)$

(*iii*) left continuous at $b$.

*A function f is said to be continuous in a semi-closed interval [a, b) if it is*

(*i*) *right continuous at a*

(*ii*) *continuous at every point of (a, b).*

*Likewise a function f is continuous in a semi-closed interval (a, b] if it is*

(*i*) *continuous at every point of (a, b)*

(*ii*) *left continuous at b.*

A function $f$ is said to be continuous on an arbitrary set $S(\subseteq R)$ if for each $\varepsilon > 0$ and $a \in S$, there exists $\delta > 0$ such that

$$| f(x) - f(a) | < \varepsilon, \text{ whenever } x \in S \text{ and } | x - a | < \delta.$$

## Examples on Continuous Functions

1. *Every constant function $f : x \to c$, is continuous on R.*

For, let $\varepsilon > 0$ be given and $a \in R$. Then,

$$| f(x) - f(a) | = | c - c | = 0 < \varepsilon,$$

whenever $| x - a | < \delta$.

2. *The identity function $f : x \to x, x \in R$ is continuous on R.*

For, let $\varepsilon > 0$ be given and $a \in R$.

Then, for $\delta = \varepsilon$, $| f(x) - f(a) | = | x - a | < \varepsilon$,

whenever $| x - a | < \delta$.

3. The function $f : x \to x^n$, $n \in N$, is continuous on R, because, for any $a \in R$,

$$\lim_{x \to a} x^n = a^n = f(a).$$

4. We know : $| \sin x | \leq | x |$ and $| \cos x | \leq 1, \forall x \in R$. Now, for any $a \in R$,

$$| \sin x - \sin a | = \left| 2 \sin \frac{x-a}{2} \cos \frac{x+a}{2} \right|$$

$$= 2 \left| \sin \frac{x-a}{2} \right| \left| \cos \frac{x+a}{2} \right| \leq 2 \left| \frac{x-a}{2} \right| = | x - a |$$

$\therefore$ given any $\varepsilon > 0$, $\exists\, \delta\ (= \varepsilon) > 0$ such that

$$|\sin x - \sin a| < \varepsilon, \text{ whenever } |x - a| < \delta$$

sin $x$ is continuous at every point $a \in$ R.

Similarly, cos $x$ is continuous on R.

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## Discontinuity of a Function

*A function f which is not continuous at a point 'a' is said to be discontinuous at the point 'a'.* '$a$' is called *the point of discontinuity of f* or $f$ is said to have a discontinuity at $a$.

*A function which is discontinuous even at a single point of an interval, is said to be discontinuous in the interval.*

A function $f$ can be discontinuous at a point $x = a$ because of anyone of the following reasons :

(*i*) $f(x)$ is not defined at $x = a$

(*ii*) $\lim_{x \to a} f(x)$ does not exist

(*iii*) $\lim_{x \to a} f(x)$ and $f(a)$ both exist but are not equal.

(*i*) If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, then we say that $f$ has a *removable discontinuity,* as just by changing the value of $f$ only at $a$ from $f(a)$ to $\lim_{x \to a} f(x)$, the new function can be made continuous at $a$.

(*ii*) If $\lim_{x \to a} f(x)$ does not exist, then the function cannot be made continuous, no matter how we define $f(a)$. In this case.

(*a*) If $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but are not equal, then we say that $f$ has a *discontinuity of first kind (or that f has a jump discontinuity at a*). It cannot be removed.

The function $f$ is said to be left discontinuous or right discontinuous at $a$ according as

$$\lim_{x \to a^-} f(x) \neq f(a) = \lim_{x \to a^+} f(x)$$

or

$$\lim_{x \to a^-} f(x) = f(a) \neq \lim_{x \to a^+} f(x)$$

(*b*) If either $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist, then we say that *f has a discontinuity of second kind.*

$f$ is said to have a discontinuity of second kind from the left or right according as $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ does not exist.

(*iii*) If a function $f$ has a discontinuity of second kind on one side of $a$ and on the other side, it may be continuous or may have discontinuity of first kind, then $f$ is said to have *mixed discontinuity* at $a$.

(*iv*) If either of the limits $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ is infinitely large, then *a is said to be a point of infinite discontinuity.*

## Examples on Discontinuities

NOTES

1. The function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous for $x \neq 2$.

*At $x = 2$* :

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

But, the value of the function is not defined at $x = 2$. Therefore, the function has a removable discontinuity. It can be made continuous at $x = 2$, by redefining it as follows :

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4 \quad , & x = 2 \end{cases}$$

2. Consider the function defined by

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ 1 \quad , & x = 0 \end{cases}$$

Now, $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{2x} \times 2 = 2$$

so that $$\lim_{x \to 0} f(x) \neq f(0).$$

Therefore, the function is discontinuous at $x = 0$. The function has a removable discontinuity at the origin as the discontinunity can be removed by redefining the function at the origin such that $f(0) = 2$.

3. Consider the function defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 0 \\ 4x + 3, & \text{if } x > 0 \end{cases}$$

Now, $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 3) = 3$$

$\therefore \quad \lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ both exist but are not equal.

Thus, the function has a discontinuity of first kind or that the function has a jump discontinuity.

4. Since in the example (3) above,

$$\lim_{x \to 0^-} f(x) = f(0) \neq \lim_{x \to 0^+} f(x)$$

therefore, the function is left continuous at 0 or the function is right discontinuous at 0.

5. Consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 4x + 3, & \text{if } x \geq 0 \end{cases}$$

Then, $\lim_{x \to 0^-} f(x) = 0, \lim_{x \to 0^+} f(x) = 3$ and $f(0) = 3$

$\therefore$ $\lim_{x \to 0^-} f(x) \neq f(0) = \lim_{x \to 0^+} f(x)$

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$\therefore$ the function has a discontinuity of first kind and the function is right continuous or left discontinuous.

6. For the function $f(x) = e^{\frac{1}{x}} \sin \frac{1}{x}$

$\lim_{x \to 0^-} f(x) = 0, \lim_{x \to 0^+} f(x)$ does not exist and the function is not defined at $x = 0$. Therefore, the function has a discontinuity of first kind from the left and a discontinuity of the second kind from the right at $x = 0$. Thus, the function has a mixed discontinuity at $x = 0$.

7. If $f(x) = \frac{1}{x-a}$, then $f(x)$ is continuous for each $x \neq a$. For $x = a$, $\lim_{x \to a^-} f(x) = -\infty$, $\lim_{x \to a^+} f(x) = \infty$ and $f(x)$ is not defined at $x = a$.

Thus, the function has an infinite discontinuity at $a$.

8. Consider the function

$$f(x) = \begin{cases} \frac{1}{x-a} \text{cosec}\,(x-a), & \text{if } x \neq a \\ 0 \quad , & \text{if } x = a \end{cases}$$

The function has infinite discontinuity at $x = a$.

For, let $x = a - h$, $(h > 0)$ so that as $x \to a^-$, $h \to 0^+$.

$$\therefore \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} \frac{1}{x-a} \text{ cosec } (x-a) = \lim_{h \to 0^+} \frac{\text{cosec}\,(a-h-a)}{a-h-a}$$

$$= \lim_{h \to 0^+} \frac{\text{cosec}\,(-h)}{-h} = \lim_{h \to 0^+} \frac{1}{h} \times \frac{1}{\sin h} = \infty$$

Since $\lim_{x \to a^-} f(x) \neq f(a)$, the function is discontinuous at $a$.

Further, one can see that $\lim_{x \to a^+} f(x) = \infty$

$\therefore$ $f$ has infinite discontinuity at $a$.

9. Consider the function

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 1 \quad , & x = 0 \end{cases}$$

in [0, 1] and in [– 1, 1].

Here, $f(x)$ is continuous everywhere on [0, 1]. However, the function defined on [– 1, 1] is not continuous at $x = 0$.

For, $\lim_{x \to 0^-} f(x) = -1, \lim_{x \to 0^+} f(x) = 1, f(0) = 1$

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$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

So, the function is left discontinuous but right continuous at 0.

10. The function

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$$

is discontinuous at $x = 0$

For, As $x \to 0^+$, $\dfrac{1}{x} \to \infty$

$\therefore \quad e^{\frac{1}{x}} \to \infty$ and $e^{-\frac{1}{x}} \to 0$

Again, when $x \to 0^-$, $\dfrac{1}{x} \to -\infty$

$\therefore \quad e^{\frac{1}{x}} \to 0$

Hence,
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \frac{0 - 1}{0 + 1} = -1$$

and
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{x \to 0^+} \frac{1 - e^{\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} = \frac{1 - 0}{1 + 0} = 1$$

$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

$\therefore \quad \lim_{x \to 0} f(x)$ does not exist.

Thus, the function is discontinuous and the discontinuity is of first kind.

## BOUNDS OF A FUNCTION

Let $f$ be a function defined on a closed interval $[a, b]$. As $x$ varies in this interval, $f(x)$ assumes varying values. If there exists a real number $k$ such that $f(x) \geq k$, $\forall\, x \notin [a, b]$, we say that the function $f$ is bounded below and $k$ is called a lower bound of $f$. If there exists a real number K such that $f(x) \leq$ K, $\forall\, x \in [a, b]$, we say that the function $f$ is bounded above and K is called an upper bound of $f$.

If there are two numbers $k$ and K such that $k \leq f(x) \leq$ K for every value of $x$ in the interval $[a, b]$, then we say that $f$ is bounded on $[a, b]$ and $k$ and K are called **lower** and **upper** bounds of $f$ over $[a, b]$.

If for any given $\varepsilon > 0$ (however small), there exists at least one $x \in [a, b]$ such that $f(x) < k + \varepsilon$ and at least one $x \in [a, b]$ such that $f(x) >$ K $- \varepsilon$, then $k$ and K are called the greatest lower bound (g.l.b.) and the least upper bound (l.u.b.) of $f$ in the interval $[a, b]$. $k$ and K are also called **supremum** and **infimum** of $f$ on $[a, b]$.

For example,

(*i*) The function sin $x$ defined on $[0, 2\pi]$ is bounded and has g.l.b. = – 1 and l.u.b. = 1.

(*ii*) The function $f$ defined on [0, 1] by

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not bounded on [0, 1].

(*iii*) The function $f$ defined by

$$f(x) = \frac{x}{x+1}, \; x \in [0, \infty)$$

is bounded and has g.l.b. = 0 and l.u.b. = 1.

## THEOREMS OF CONTINUOUS FUNCTIONS

### Theorem

*A function f is continuous at a if and only if for every sequence $< x_n >$ converging to a ($x_n \in D$, domain of the function f), the sequence $< f(x_n) >$ converges to f(a).*

**Proof.** (*i*) Let $f$ be continuous at $a$ and the sequence $< x_n >$ converges to $a$.

Since $f$ is continuous at $a$.

$\therefore$ given any $\varepsilon > 0$, $\exists\, \delta > 0$, such that

$$| f(x) - f(a) | < \varepsilon, \text{ whenever } | x - a | < \delta \qquad ...(1)$$

Since $< x_n >$ converges to $a$, $\exists$ a positive integer $m$ such that

$$| x_n - a | < \delta, \quad \forall\, n \geq m \qquad ...(2)$$

From (1) and (2), we have

$$| f(x_n) - f(a) | < \varepsilon \quad \forall\, n \geq m$$

$\Rightarrow$ $< f(x_n) >$ converges to $f(a)$.

(*ii*) We are given that whenever $x_n \to a$, then $< f(x_n) >$ converges to $f(a)$. We are to show that $f$ is continuous at $a$.

Let if possible, $f$ be not continuous at $a$. Then, $\exists\, \varepsilon > 0$, such that for every $\delta > 0$, there exists a point $x$ such that

$$| x - a | < \delta \text{ and } | f(x) - f(a) | \geq \varepsilon$$

Taking $\delta = \frac{1}{n}$, we can find a point $x_n$ such that

$$| x_n - a | < \frac{1}{n} \text{ and } | f(x_n) - f(a) | \geq \varepsilon \quad \forall\, n$$

$\Rightarrow$ $< x_n >$ converges to $a$ and $< f(x_n) >$ does not converge to $f(a)$. This contradiction proves that $f$ is continuous at $x = a$.

**Remark.** *The choice of $< x_n >$ in the δ-neighbourhood of a such that $< x_n >$ converges to a is infinite. For example, $x_n = a + \frac{\delta_1}{2^n}$ where $0 < \delta_1 < \delta$, is such a choice.*

## Theorem

*If f and g are two continuous functions at a, then*

(*i*) *$f + g$ is continuous at a* (*ii*) *$f - g$ is continuous at a*

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(*iii*) *fg is continuous at a.* (*iv*) *$\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$*

(*v*) *cf is continuous at a, where c is constant.*

**Proof.** Let $< a_n >$ be a sequence converging to $a$.

(*i*) Since $f$ is continuous at $a$, $< f(a_n) >$ converges to $f(a)$.

Since $g$ is continuous at $a$, $< g(a_n) >$ converges to $g(a)$.

$$\therefore \quad \lim_{n\to\infty} (f+g)(a_n) = \lim_{n\to\infty} (f(a_n) + g(a_n))$$

$$= \lim_{n\to\infty} f(a_n) + \lim_{n\to\infty} g(a_n) = f(a) + g(a) = (f+g)(a)$$

$\Rightarrow$ the sequence $< (f+g)(a_n) >$ converges to $(f+g)(a)$.

$\Rightarrow f + g$ is continuous at $a$.

(*ii*) As of part (*i*).

(*iii*) Since $f$ is continuous at $a$, $< f(a_n) >$ converges to $f(a)$.

Since $g$ is continuous at $a$, $< g(a_n) >$ converges to $g(a)$.

$$\therefore \quad \lim_{n\to\infty} (fg)(a_n) = \lim_{n\to\infty} (f(a_n)\, g(a_n))$$

$$= \lim_{n\to\infty} f(a_n) \lim_{n\to\infty} g(a_n) = f(a)\, g(a) = (fg)(a)$$

$\Rightarrow$ the sequence $< (fg)(a_n) >$ converges to $(fg)(a)$.

$\Rightarrow$ $fg$ is continuous at $a$.

(*iv*) Since $f$ is continuous at $a$, $< f(a_n) >$ converges to $f(a)$.

Since $g$ is continuous at $a$, $< g(a_n) >$ converges to $g(a)$.

Since $g(a) \neq 0$, there exists a positive integer $m$ such that $g(a_n) \neq 0$, $\forall\ n \geq m$.

$$\therefore \quad \lim_{n\to\infty} \left(\frac{f}{g}\right)(a_n) = \lim_{n\to\infty} \left(\frac{f(a_n)}{g(a_n)}\right) = \frac{\lim_{n\to\infty} f(a_n)}{\lim_{n\to\infty} g(a_n)} = \frac{f(a)}{g(a)}$$

$\Rightarrow$ the sequence $< \left(\frac{f}{g}\right)(a_n) >$ converges to $\left(\frac{f}{g}\right)(a)$.

$\Rightarrow$ $\frac{f}{g}$ is continuous at $a$.

(*v*) Left to the reader as an exercise.

**Remark.** *Since a function f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$, the proofs of the above theorem can also be deduced from the corresponding results on limits.*

**Note.** (1) Every polynomial function is continuous for every real number.

(2) Every rational function is continuous for every real number other than zeros of the denominator.

## Theorem

*If a function f is continuous at a, then | f | is also continuous at a.*

**Proof.** Let $\varepsilon > 0$ be given.

Since $f$ is continuous at $a$, therefore, there exists $\delta > 0$ such that

$$| f(x) - f(a) | < \varepsilon, \text{ whenever } | x - a | < \delta$$

Now, $| \, | f(x) | - | f(a) | \, | \leq | f(x) - f(a) | < \varepsilon$, whenever $| x - a | < \delta$

$$(\because \; | a - b | \geq | \, | a | - | b | \, | )$$

$\Rightarrow | f |$ is continuous at $a$.

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**Remark.** Converse of the above theorem is false. For example, if $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$ and $a$ is any real number, then in the interval $(a - d, a)$, there lie infinitely many rationals as well as irrationals. Therefore, $f(x)$ oscillates between $-1$ and 1. Therefore, $f$ cannot tend to a definite value as $x \to a^-$.

Thus $\lim\limits_{x \to a^-} f(x)$ does not exist.

Similarly, $\lim\limits_{x \to a^+} f(x)$ does not exist.

Hence $f$ is not continuous at $a$.

But $| f(x) | = 1 \; \forall \; x \in R$

Hence $| f |$ is continuous everywhere and in particular at $a$ also.

**Cor.** *If f and g are two continuous functions at a, then the functions max. {f, g} and min. {f, g} are both continuous at a.*

For, $$\max. \{f, g\} = \frac{1}{2} (f + g) + \frac{1}{2} | f - g |$$

and $$\min. \{f, g\} = \frac{1}{2} (f + g) - \frac{1}{2} | f - g |.$$

## Theorem

*Composite of two continuous functions is a continuous function, i.e., if f is a continuous function at x = a and g is a continuous function at f(a), then gof is a continuous function at x = a.*

**Proof.** Let $\varepsilon > 0$ be given.

Since $g$ is continuous at $f(a)$

$\therefore \; \exists \; \delta > 0$ such that

$$| g(f(x)) - g(f(a)) | < \varepsilon \text{ whenever } | f(x) - f(a) | < \delta \quad ...(1)$$

Since $f$ is continuous at $x = a$,

$\therefore \; \exists \; \delta_1 > 0$ (depending upon $\delta$) such that

$$| f(x) - f(a) | < \delta \text{ whenever } | x - a | < \delta_1 \quad ...(2)$$

$\therefore$ from (1) and (2), we have, for any $\varepsilon > 0$, $\exists \; \delta_1 > 0$ such that

$$| (gof)(x) - (gof)(a) | < \varepsilon \text{ whenever } | x - a | < \delta_1$$

$\Rightarrow$ $gof$ is continuous at $x = a$.

### Theorem

*A function f is continuous at a if and only if for each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $| f(x_1) - f(x_2) | < \varepsilon$*, whenever* $x_1, x_2 \in (a - \delta, a + \delta)$*.*

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**Proof.** Let $f$ be continuous at $a$.

Then, for $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$| f(x) - f(a) | < \frac{\varepsilon}{2}, \text{ whenever } | x - a | < \delta.$$

$\therefore$ for $x_1, x_2 \in (a - \delta, a + \delta)$,

$$\begin{aligned} | f(x_1) - f(x_2) | &= | f(x_1) - f(a) + f(a) - f(x_2) | \\ &\le | f(x_1) - f(a) | + | f(x_2) - f(a) | \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Conversely, let for each $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$| f(x_1) - f(x_2) | < \varepsilon, \text{ whenever } x_1, x_2 \in (a - \delta, a + \delta).$$

Taking $x_1 = x$ and $x_2 = a$, we have

$$| f(x) - f(a) | < \varepsilon, \text{ whenever } x \in (a - \delta, a + \delta)$$

$\Rightarrow$ $f$ is continuous at $a$.

### Theorem

*If a function f is continuous at a, then it is bounded in some neighbourhood of a, i.e., there exists* $\delta > 0$ *and* $k > 0$ *such that* $| f(x) | \le k$*, for every x in* $(a - \delta, a + \delta)$ *at which f is defined.*

**Proof.** Take $\varepsilon = 1$. Since $f$ is continuous at $a$, $\exists\ \delta > 0$, such that

$$| f(x) - f(a) | < 1, \text{ whenever } | x - a | < \delta, x \in D_f$$

$\Rightarrow$ $f(a) - 1 < f(x) < f(a) + 1$, whenever $| x - a | < \delta$

$\Rightarrow$ $f(x)$ is bounded.

### Theorem

*Every function continuous on a closed interval is bounded in that interval.*

**Proof.** Let $f$ be a continuous function on I ( = $[a, b]$). Let $f$ be not bounded above in $[a, b]$. Let $c$ be the mid-point of $[a, b]$ $\left(i.e., c = \frac{a + b}{2}\right)$. Then $f$ must be unbounded above in at least one of the intervals $[a, c]$ and $[c, b]$. Let us call that interval $I_1$ and rewrite it as $[a_1, b_1]$ (if $f$ is unbounded in both the subintervals, then we take $[a, c]$ as $[a_1, b_1]$). Let $c_1$ be the mid-point of $[a_1, b_1]$. Then $f$ must be unbounded above in at least one of the intervals $[a_1, c_1]$ and $[c_1, b_1]$. Call that interval $I_2$ and rewrite it as $[a_2, b_2]$. (If $f$ is unbounded in both the intervals, then we call $[a_1, c_1]$ as $[a_2, b_2]$). Continuing like this, we get a nested sequence of closed intervals $I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$. such that

(*i*) length $(I_n) = b_n - a_n = \left(\frac{1}{2^n}\right)(b - a)$ so that

$$\lim_{n \to \infty} (\text{length } I_n) = 0$$

(*ii*) $f$ is unbounded in each of the intervals $I_n$.

By Nested Interval Property $\bigcap_{n=1}^{\infty} I_n$ is a singleton = {ξ}, say. Since $f$ is continuous on I, $f$ is continuous at ξ also. Therefore, given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$| f(x) - f(\xi) | < \varepsilon \text{ whenever } | x - \xi | < \delta.$$

Since $\lim_{n \to \infty}$ (length of $I_n$) = 0, ∃ a positive integer $m$ such that $I_m \subseteq (\xi - \delta, \xi + \delta)$. If $x$ is any point of $I_m$, then

$$| f(x) | = | f(x) - f(\xi) + f(\xi) |$$
$$\leq | f(x) - f(\xi) | + | f(\xi) | < \varepsilon + | f(\xi) |,$$

showing that $f$ is bounded in $I_m$. This contradicts the fact that $f$ is unbounded above in each subinterval $I_n$. Hence $f$ must be bounded above in I. Similarly, $f$ is bounded below in $[a, b]$. Hence $f$ is bounded in $[a, b]$.

**Remark.** The above theorem need not be true if the interval is not closed.

For example :

(*i*) Let $f(x) = \frac{1}{x}$, $x \in (0, 1]$.

Then $f$ is continuous on (0, 1].

As $x \to 0^+$, $\frac{1}{x} \to \infty$. Therefore, $f$ is not bounded on (0, 1].

(*ii*) $f(x) = \log x$, $x \in (0, \infty)$

$f$ is continuous on $(0, \infty)$, but it is not bounded.

## Theorem

*Every function continuous on a closed interval attains its bounds (at least once on the interval).*

**Proof.** Let $f$ be a continuous function defined on a closed interval I.

∴ $f$ is bounded.

∴ l.u.b. and g.l.b. of $f$ exist.

Let $u$ = l.u.b. of $f$ and $l$ = g.l.b. of $f$.

We shall show that there exist real numbers ξ and η in I such that $f(\xi) = u$ and $f(\eta) = l$. Let, if possible, $f(x) < u\ \forall\ x \in I$.

$$\Rightarrow \quad u - f(x) > 0 \quad \forall\ x \in I.$$

Since $f$ is continuous on I, $\frac{1}{u - f(x)}$ is also continuous on I

$\Rightarrow$ $\frac{1}{u - f(x)}$ is bounded.

∴ ∃ a positive number $k$ such that

$$\frac{1}{u - f(x)} \leq k \quad \forall\ x \in I \quad \Rightarrow \quad f(x) \leq u - \frac{1}{k} \quad \forall\ x \in I$$

$\Rightarrow$ $u - \frac{1}{k}$ is an upper bound of $f$

This contradicts the fact that $u$ is the l.u.b. of $f$. Hence there must exist some $\xi \in I$ such that

$$u - f(\xi) = 0, \text{ i.e., } f(\xi) = u.$$

Similarly, there must exist at least one point $\eta \in I$ such that $f(\eta) = l$.

**Remark.** The above theorem need not be true if the interval is not closed. For example :

1. The function $f$ defined on (0, 1] or ([0, 1) or (0, 1)) by $f(x) = x$ is continuous and bounded on (0, 1] or ([0, 1) or (0, 1)) with g.l.b. of $f = 0$ and l.u.b. of $f = 1$. g.l.b. of $f$ is not attained on (0, 1], l.u.b. of $f$ is not attained on [0, 1) and neither is attained on (0, 1).

2. The function $y = x$ is continuous on R but is not bounded on any infinite interval.

3. The function $f(x) = \dfrac{x^2}{x^2 + 1}$ is continuous and bounded on R. It attains its g.l.b. = 0 but not the l.u.b. = 1 on any interval. Here $f(x)$ is not continuous on the closed interval (R is open). So, the above theorem is not applicable.

## Theorem

*Let $f$ be a continuous function on $[a, b]$ and $a < c < b$. Then*

(*i*) *$f(a) > 0$ ($f(a) < 0$) implies that there exists $\delta > 0$ such that $f(x) > 0$ ($< 0$) for all $x$ in $[a, a + \delta)$.*

(*ii*) *$f(c) > 0$ ($f(c) < 0$) implies that there exists $\delta > 0$ such that $f(x) > 0$ ($f(x) < 0$) for all $x$ in $(c - \delta, c + \delta)$.*

(*iii*) *$f(b) > 0$ ($f(b) < 0$) implies that there exists $\delta > 0$ such that $f(x) > 0$ ($f(x) < 0$) for all $x$ in $(b - \delta, b]$.*

**Proof.** (*i*) *Let $f(a) > 0$*. Since $f$ is right continuous at $a$, corresponding to any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$| f(x) - f(a) | < \varepsilon, x \in [a, a + \delta)$$

$$\Rightarrow \quad f(a) - \varepsilon < f(x) < f(a) + \varepsilon, x \in [a, a + \delta)$$

Taking $\varepsilon = f(a)$, $\quad 0 < f(x) < 2f(a), x \in [a, a + \delta)$

$$\Rightarrow \quad f(x) > 0. \qquad (\because\ f(a) > 0)$$

Proof for the case $f(a) < 0$ is similar.

(*ii*) *Let $f(c) > 0$.*

Take $\quad \varepsilon = \dfrac{f(c)}{2} > 0$

Then $\exists\ \delta > 0$ such that

$$| f(x) - f(c) | < \varepsilon, \ | x - c | < \delta$$

$$\Rightarrow \quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \ | x - c | < \delta$$

$$\Rightarrow \quad f(c) - \tfrac{1}{2} f(c) < f(x) < f(c) + \tfrac{1}{2} f(c), \ | x - c | < \delta$$

$$\Rightarrow \quad f(x) > \tfrac{1}{2} f(c) > 0, \ | x - c | < \delta$$

Proof for the case $f(c) < 0$ is similar.

(*iii*) Proof is left for the reader as an exercise.

**Remark.** The above theorem asserts that if a function is continuous at a point, then its sign is invariable near the point.

## Theorem

*Let $f$ be a continuous function on R. Then,*

(*i*) *the set $A = [x \in R \mid f(x) > 0]$ is an open set*

(*ii*) *the set $B = [x \in R \mid f(x) < 0]$ is an open set*

(*iii*) *the set $C = [x \in R \mid f(x) = 0]$ is a closed set.*

**Proof.** (*i*) Let $a \in A$.

$$\Rightarrow \quad f(a) > 0.$$

NOTES

Since $f$ is continuous at $a$, there exists $\delta > 0$ such that

$$f(x) > 0 \text{ for } x \in (a - \delta, a + \delta)$$

$\Rightarrow$ $$x \in (a - \delta, a + \delta) \subseteq A$$

$\Rightarrow$ A is a neighbourhood of $a$.

Since $a$ is arbitrary, this implies that A is an open set.

(*ii*) Proof is similar to (*i*).

(*iii*) Since $C = R - (A \cup B)$, it follows that C is a closed set.

**Cor.** *Let f be a continuous function on R and c any real number. Then,*

(*i*) *the set* $A = \{x \in R \mid f(x) < c\}$ *is an open set*

(*ii*) *the set* $B = \{x \in R \mid f(x) > c\}$ *is an open set*

(*iii*) *the set* $C = \{x \in R \mid f(x) = c\}$ *is a closed set.*

**Proof.** (*i*) Define a function $g$ as :

$$g(x) = f(x) - c, \quad \forall\, x \in R.$$

Since $f$ is continuous on R, $g$ is continuous on R.

$\therefore$ the set $\{x \in R \mid g(x) < 0\}$ is an open set

$\Rightarrow$ the set $\{x \in R \mid f(x) - c < 0\}$ is an open set

$\Rightarrow$ the set $A = \{x \in R \mid f(x) < c\}$ is an open set

Proofs of (*ii*) and (*iii*) are on similar lines.

## Theorem

*A function* $f : R \longrightarrow R$ *is continuous on R if and only if for every open set A in R,* $f^{-1}(A)$ *is open in R.*

**Proof.** Let $f$ be continuous on R.

Let A be an open subset of R.

If $f^{-1}(A) = \phi$, then it is open

So, let $$f^{-1}(A) \neq \phi$$

Let $$a \in f^{-1}(A) \quad \Rightarrow \quad f(a) \in A$$

Since A is open, A is a neighbourhood of each of its points.

$\therefore$ $\exists\, \varepsilon > 0$ such that $(f(a) - \varepsilon, f(a) + \varepsilon) \subseteq A$.

Since $f$ is continuous at $a$, (for above $\varepsilon > 0$) $\exists\, \delta > 0$ such that

$| f(x) - f(a) | < \varepsilon$, for $| x - a | < \delta$ *i.e.*, $x \in (a - \delta, a + \delta)$

Hence, $x \in (a - \delta, a + \delta) \quad \Rightarrow \quad f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon) \subseteq A$

$\Rightarrow$ $x \in f^{-1}(A)$

$\Rightarrow$ $a \in (a - \delta, a + \delta) \subseteq f^{-1}(A)$

$\Rightarrow$ $f^{-1}(A)$ is a neighbourhood of each of its points

$\Rightarrow$ $f^{-1}(A)$ is open.

Conversely, let $f$ be not continuous at $a$.

Then, there exists an $\varepsilon > 0$, such that for any $\delta > 0$, there exists $x_0$ such that $| x_0 - a | < \delta$ but

$$| f(x_0) - f(a) | \nless \varepsilon \quad i.e., \quad f(x_0) \notin (f(a) - \varepsilon, f(a) + \varepsilon).$$

Therefore, every open interval $(a - \delta, a + \delta)$ containing $a$, contains an $x_0$ such that $f(x_0) \notin (f(a) - \varepsilon, f(a) + \varepsilon)$

$\Rightarrow f^{-1}(f(a) - \varepsilon, f(a) + \varepsilon)$ is not an open set. This contradicts the given hypothesis ($\because$ $(f(a) - \varepsilon, f(a) + \varepsilon)$ is open). Hence, $f$ is continuous at $a$. Since $a$ is arbitrary, $f$ is continuous at every point of R.

**Cor.** *A function $f: R \longrightarrow R$ is continuous on R if and only if for every closed set A in R, $f^{-1}(A)$ is closed in R.*



**Proof.** Let $f$ be continuous on R.

Let A be a closed subset of R

$\Rightarrow$ $A^c$ is an open subset of R

$\Rightarrow$ $f^{-1}(A^c)$ is open

$\Rightarrow$ $(f^{-1}(A))^c$ is open

$\Rightarrow$ $f^{-1}(A)$ is closed.

Conversely, for a closed set A, let $f^{-1}(A)$ be closed.

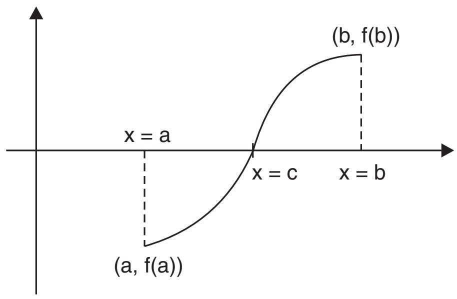
$\Rightarrow$ $A^c$ and $(f^{-1}(A))^c$ are open

$\Rightarrow$ $A^c$ and $f^{-1}(A^c)$ are open

$\Rightarrow$ $f$ is continuous on R.

## Theorem

*If a function $f$ is continuous on a closed interval $[a, b]$, $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point $c \in (a, b)$ such that $f(c) = 0$.*




**Proof.** By considering, if necessary, the function $-f$, we can assume without loss of generality that $f(a) < 0$ and $f(b) > 0$.

Let $S = \{x : x \in [a, b] \text{ and } f(x) < 0\}$,

$$S \neq \phi \qquad (\because\ a \in S)$$

Also S is bounded above $(\because\ b = \text{u.b. } S)$

$\therefore$ by completeness property of reals, S has the l.u.b. $c$ (say). We shall show that $f(c) = 0$. Suppose that $f(c) < 0$.

Then $c \neq b$ $(\because\ f(b) > 0)$ and by the continuity of $f$ at $c$, $\exists\ \delta > 0$ such that

$$f(x) < 0,\ x \in [c, c + \delta).$$

$$\Rightarrow \qquad f\left(c + \frac{\delta}{2}\right) < 0.$$

$\Rightarrow$ $c + \dfrac{\delta}{2} \in S$, which contradicts the fact that

$$c = \text{l.u.b. } S.$$

Hence, $f(c) \nless 0$.

Suppose that $f(c) > 0$. Then $c \neq a$ $\qquad (\because\ f(a) < 0)$

By continuity of $f$ at $c$, $\exists\ \delta > 0$ such that

$$f(x) > 0 \text{ for } x \in (c - \delta, c].$$

Since $c = \text{l.u.b. } S$, there exists $t \in S$ such that $c - \delta < t < c$.

Now $\quad t \in S \Rightarrow f(t) < 0$

Since $\quad t \in (c - \delta, c]$, $f(t) > 0$.

This gives a contradiction. Hence $f(c) = 0$ and $c \neq a$, $c \neq b$.

**Cor. 1.** (*Intermediate Value Theorem*)

*If a function $f$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then $f$ assumes every value between $f(a)$ and $f(b)$.*

**Proof.** W.l.o.g, we assume that $f(a) < f(b)$. Let $k$ be any number such that

$$f(a) < k < f(b).$$

Consider a function $g$ defined on $[a, b]$ such that

$$g(x) = f(x) - k.$$

Now $g$ is continuous on $[a, b]$ and $g(a) = f(a) - k$,

$g(b) = f(b) - k$ are of opposite signs.

Therefore, $\exists\ c \in (a, b)$ such that $g(c) = 0$, *i.e.*, $f(c) = k$.

**Remark.** Converse of Intermediate Value Theorem need not be true. For example,

Let $$f(x) = \begin{cases} x, & 0 \le x < 1 \\ x - 1, & 1 \le x \le 3 \end{cases}$$

Then $f(x)$ assumes every value between $f(0) = 0$ and $f(3) = 2$ as $x$ moves through 0 to 3 but $f$ is not continuous on $[0, 3]$.

**Cor. 2.** *A function f, which is continuous on a closed interval [a, b], assumes every value between its bounds.*

**Proof.** Since the function $f$ is continuous on $[a, b]$, it is bounded and attains its bounds on $[a, b]$.

$\therefore\quad \exists$ two numbers $\alpha$ and $\beta$ in $[a, b]$ such that $f(\alpha) = M$ and $f(\beta) = m$, when M and $m$ are the l.u.b. and the g.l.b. of $f$.

Since $f$ is continuous on $[a, b]$, it is continuous on $[\beta, \alpha]$ or $[\alpha, \beta]$ (according as $\alpha > \beta$ and $\alpha < \beta$).

Hence, $f$ assumes every value between $f(\alpha)$ and $f(\beta)$ (*i.e.*, the function assumes every value between its bounds).

*We also say that the range of a continuous function on a closed interval, is a closed interval, or that the image of a closed interval under a continuous function is a closed interval.*

## MONOTONIC FUNCTIONS

Let $f$ be a function defined on $[a, b]$. Then,

(*i*) the function $f$ is said to be monotonically increasing on $[a, b]$, if for $x_1, x_2 \in [a, b]$, $x_1 > x_2$ implies $f(x_1) \ge f(x_2)$.

(*ii*) the function $f$ is said to be monotonically decreasing on $[a, b]$ if for $x_1, x_2 \in [a, b]$, $x_1 > x_2$ implies $f(x_1) \le f(x_2)$.

(*iii*) the function $f$ is said to be strictly monotonically increasing on $[a, b]$ if for $x_1, x_2 \in [a, b]$, $x_1 > x_2$ implies $f(x_1) > f(x_2)$.

(*iv*) the function, $f$ is said to be strictly monotonically decreasing on $[a, b]$ if for $x_1, x_2 \in [a, b]$, $x_1 > x_2$ implies $f(x_1) < f(x_2)$.

(*v*) the function $f$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

### Theorem

*(Continuity of inverse function). If a one to one function f is continuous and strictly monotonic on [a, b], then* $f^{-1}$ *is also continuous on [f(a), f(b)] or [f(b), f(a)] according as f is increasing or decreasing.*

**Proof.** Since the function $f$ is continuous and strictly monotonic on $[a, b]$, $f$ is bounded and attains its bounds at $a$ and $b$.

$\therefore$ $\exists$ real numbers $m$ and M such that

$$f(a) = m,\ f(b) = M \quad \text{or} \quad f(a) = M,\ f(b) = m$$

according as $f$ is strictly monotonically increasing or decreasing.

$\Rightarrow$ range of $f$ is $[f(a), f(b)]$ or $[f(b), f(a)]$

according as $f$ is strictly monotonically increasing or decreasing.

Since $f$ is one to one, $f^{-1}$ exists. $\therefore$ $f^{-1}$ is a function with domain $[f(a), f(b)]$ or $[f(b), f(a)]$ according as $f$ is strictly monotonically increasing or decreasing.

Let $f$ be strictly monotonically increasing.

$\Rightarrow$ $f^{-1}$ is a function with domain $[f(a), f(b)]$.

We shall show that $f^{-1}$ is continuous on $[f(a), f(b)]$.

Let $\quad y_0 \in [f(a), f(b)]$

$\Rightarrow \exists$ some $x_0 \in [a, b]$ such that $f^{-1}(y_0) = x_0$, *i.e.*, $f(x_0) = y_0$

Let $\varepsilon > 0$ be given

Let $\quad f(x_0 - \varepsilon) = y_0 - \delta_1$ and $f(x_0 + \varepsilon) = y_0 + \delta_2$,

where $\delta_1, \delta_2$ are necessarily positive numbers.

Since $f$ is strictly monotonically increasing,

$$x \in (x_0 - \varepsilon, x_0 + \varepsilon) \Rightarrow f(x) \in [f(x_0 - \varepsilon), f(x_0 + \varepsilon)]$$

*i.e.*, $\quad f(x) \in [y_0 - \delta_1, y_0 + \delta_2]$

If $\delta = \min \{\delta_1, \delta_2\}$, then $(y_0 - \delta, y_0 + \delta) \subseteq (f(x_0 - \varepsilon), f(x_0 + \varepsilon))$

$\therefore \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for $y = f(x) \in (y_0 - \delta, y_0 + \delta)$

*i.e.*, $\quad | x - x_0 | < \varepsilon$ for $| y - y_0 | < \delta$

*i.e.*, $\quad | f^{-1}(y) - f^{-1}(y_0) | < \varepsilon$ for $| y - y_0 | < \delta$

$\Rightarrow f^{-1}$ is continuous at $y_0 \in [f(a), f(b)]$

Since $y_0$ is arbitrary, $f^{-1}$ is continuous at every point of $[f(a), f(b)]$ and hence $f^{-1}$ is continuous on $[f(a), f(b)]$.

Similar is the case when $f$ is strictly monotonically decreasing.

## UNIFORM CONTINUITY

Recall that if a function $f$ is continuous at a point $x_0$ of the closed interval $[a, b]$, then given any $\varepsilon > 0$, there exists *a* $\delta > 0$ such that $| f(x) - f(x_0) | < \varepsilon$ whenever $| x - x_0 | < \delta$. This $\delta$ depends not only on $\varepsilon$ but also on the point $x_0$ at which the continuity is considered. That is, if $\varepsilon > 0$ remains the same for different points of the interval $[a, b]$, then the choice of $\delta$ may not be the same. If it is possible to find one $\delta > 0$ depending on $\varepsilon$ such that $| f(x) - f(x_0) | < \varepsilon$ whenever $| x - x_0 | < \delta$ and $x, x_0 \in [a, b]$, then we say that the function in uniformly continuous. More precisely.

**Definition.** *A function f is said to be uniformly continuous on the interval [a, b] if given any* $\varepsilon > 0$*, there exists* $\delta > 0$ *(depending on* $\varepsilon$ *only) such that* $| f(x_1) - f(x_2) | < \varepsilon$, *whenever* $x_1, x_2 \in [a, b]$ *and* $| x_1 - x_2 | < \delta$.

**Remarks 1.** The notion of uniform continuity of a function is a global concept. It does not make sense to say that a function is uniformly continuous at a point, while continuity is a local property.

**2.** One may think that the g.l.b. of the set of all values of δ corresponding to different points of $[a, b]$, would serve the purpose of δ for uniform continuity on $[a, b]$. But it is not so as the g.l.b. of a set of positive numbers may be zero.

**3.** A function $f$ is not uniformly continuous on $[a, b]$ if ∃ some $\varepsilon > 0$ for which no δ works, *i.e.,* for any $\delta > 0$, $\exists\, x_1, x_2 \in [a, b]$ such that $| f(x_1) - f(x_2) | \geq \varepsilon$ and $| x_1 - x_2 | < \delta$.

## Theorem

*If a function f is uniformly continuous on [a, b], then it is continuous on [a, b].*

**Proof.** Let $f$ be uniformly continuous on $[a, b]$

Let $\varepsilon > 0$ be given.

Since $f$ is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$| f(x_1) - f(x_2) | < \varepsilon \quad \forall\ x_1, x_2 \in [a, b] \quad \text{and} \quad |x_1 - x_2| < \delta.$$

Let $\qquad c \in [a, b]$.

Taking $x_1 = x$ and $x_2 = c$, we have for $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$| f(x) - f(c) | < \varepsilon, \text{ for } | x - c | < \delta$$

$\Rightarrow f$ is continuous at $c$, any point of $[a, b]$. $\Rightarrow f$ is continuous at every point of $[a, b]$.

**Note.** In fact, we can consider any interval open or semi-open in the above theorem.

**Remark.** Converse of the above theorem need not be true. For example, consider the function $f$ defined by

$$f(x) = \frac{1}{x}, \quad x \in (0, 1)$$

Since $x$ is continuous on $(0, 1)$ and $x \neq 0$, therefore $\frac{1}{x}$ is continuous on $(0, 1)$.

Again, for any $\delta > 0$, $\exists\ m \in N$ such that $\frac{1}{n} < \delta \ \ \forall\ n \geq m$.

Let $\qquad x_1 = \frac{1}{2m}$ and $x_2 = \frac{1}{m}$ so that $x_1, x_2 \in (0, 1)$,

$$| x_1 - x_2 | = \left| \frac{1}{2m} - \frac{1}{m} \right| = \frac{1}{2m} < \frac{1}{2}\delta < \delta.$$

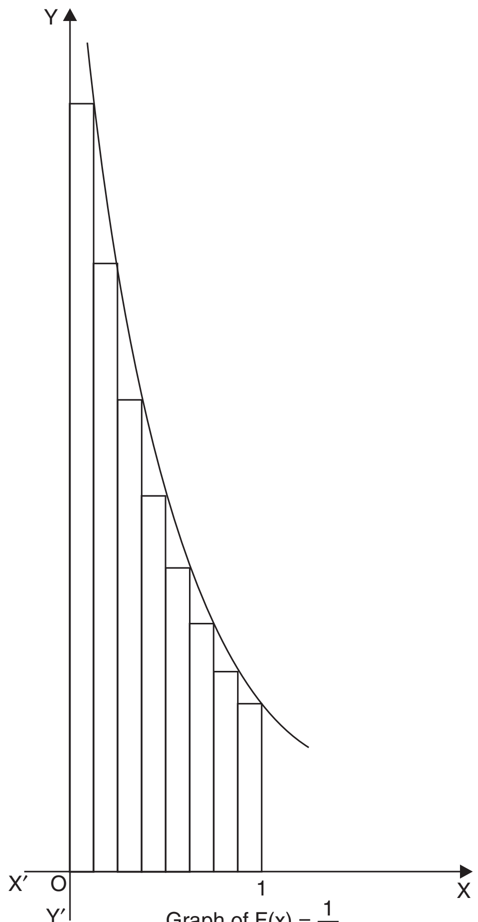
and $\qquad | f(x_1) - f(x_2) | = | 2m - m | = m$

which cannot be less than every $\varepsilon > 0$.

Hence $f$ is not uniformly continuous on $(0, 1)$. This can also be seen geometrically as follows :

Consider the graph of the function $f(x) = \frac{1}{x}$ on $(0, 1)$. If we divide the interval $(0, 1)$ into subintervals of equal width, then the increment in the values of $f(x)$ are not equal, showing thereby that the function is not uniformly continuous on $(0, 1)$.

NOTES




Graph of $F(x) = \frac{1}{x}$

## Theorem

*If a function f is continuous on [a, b], then f is uniformly continuous on [a, b].*

To prove the above theorem, we first prove the following Lemma :

## Lemma

**If a function f : [a, b] → R is continuous on [a, b], then for given ε > 0, [a, b] can be divided into a finite number of sub-intervals such that | $f(x_1) - f(x_2)$ | < ε whenever $x_1$ and $x_2$ lie in the same sub-interval.**

**Proof of Lemma :**

Let if possible, the result be false. Then, there exists none Σ > 0, such that for no subdivision of [$a$, $b$], | $f(x_1) - f(x_2)$ | < ε for $x_1$, $x_2$ in the same sub-interval.

Divide [$a$, $b$] into two equal sub-intervals. Then, the result is not satisfied in at least one of the two sub-intervals. Call the sub-interval [$a_1$, $b_1$] in which the result is not satisfied. Again divide [$a_1$, $b_1$] into two equal sub-intervals and then the result is not satisfied in at least one of the two sub-intervals, say [$a_2$, $b_2$] and so on. In this way, we get a nested sequence of closed intervals with lengths :

$$b_1 - a_1 = \frac{1}{2}(b-a),\ b_2 - a_2 = \frac{1}{2^2}(b-a), \text{...... and so on.}$$

$$b_n - a_n = \frac{1}{2^n}(b-a) \to 0 \text{ as } n \to \infty$$

Therefore, by Nested Interval Property,

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{x_0\} \text{ . (say) . and } x_0 \in [a, b].$$

**Case** (*i*) $x_0 \in (a, b)$

Since $f$ is continuous on $[a, b]$

$f$ is continuous at $x_0$ also.

$\therefore\ \exists\ \delta > 0$ such that

$$| f(x) - f(x_0) | < \frac{\varepsilon}{2}, \ \forall x \in (x_0 - \delta, x_0 + \delta)$$

Also, $b_n - a_n \to 0$ as $n \to \infty$

$\therefore$ We can choose a + ve integer $n$ such that $b_n - a_n < \delta$.

$$(a_n, b_n) \subseteq (x_0 - \delta, x_0 + \delta).$$

$$\therefore \qquad | f(x) - f(x_0) | < \frac{\varepsilon}{2}, \ \forall\ x \in (a_n, b_n).$$

Let $x_1, x_2 \in (a_n, b_n)$

$$\therefore \qquad | f(x_1) - f(x_0) | < \frac{\varepsilon}{2} \text{ and } | f(x_2) - f(x_0) | < \frac{\varepsilon}{2}.$$

$$\therefore \qquad | f(x_1) - f(x_2) | = | f(x_1) - f(x_0) + f(x_0) - f(x_2) |$$

$$\leq | f(x_1) - f(x_0) | + | f(x_2) - f(x_0) | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore$ the result is satisfied in $(a_n, b_n)$

which is a contradiction to our supposition.

Hence the Lemma is true.

**Case** (*ii*) $x_0 = a$ or $x_0 = b$

If $x_0 = a$, then the result follows from above by taking the interval $(a, a + \delta)$.

If $x_0 = b$, then the result follows by taking the interval $(b - \delta, b)$.

**Proof of Theorem.** Let $\varepsilon > 0$ be given.

Since it is continuous on $[a, b]$, therefore by the Lemma, the interval $[a, b]$ can be divided into a finite number of sub-intervals

$[a, y_1], [y_1, y_2], ......, [y_n, b]$, (say) such that whenever $x_1, x_2$ are in the same sub-interval,

$$| f(x_1) - f(x_2) | < \frac{\varepsilon}{2} \qquad ...(1)$$

Let $\delta = \min. \{y_1 - a, y_2 - y_1, ......, b - y_n\} > 0$

Now, if $x_1, x_2 \in [a, b]$ with $| x_1 - x_2 | < \delta$ ...(2)

then either $x_1, x_2$ belong to the same sub-interval or they belong to two consecutive sub-intervals.

**Case** (*i*) $\mathbf{x_1, x_2}$ **belong to the same sub-interval**

$$| f(x_1) - f(x_2) | < \frac{\varepsilon}{2} < \varepsilon \qquad \text{[By (1)]}$$

**Case** (*ii*) $\mathbf{x_1, x_2}$ **lie in consecutive intervals**

Let $y_i$ be the point of division of two sub-intervals in which $x_1$ and $x_2$ lie.

$\therefore$ $x_1, y_i$ lie in the same sub-intervals and $y_i, x_2$ lie in the same sub-interval.

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$\therefore \quad | f(x_1) - f(y_i) | < \frac{\varepsilon}{2}$ ...(3)

and $\quad | f(y_i) - f(x_2) | < \frac{\varepsilon}{2}$ ...(4)

Now, 
$$\begin{aligned} | f(x_1) - f(x_2) | &= | [f(x_1) - f(y_i)] + [f(y_i) - f(x_2)] | \\ &\le | f(x_1) - f(y_i) | + | f(y_i) - f(x_2) | \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $x_1, x_2 \in [a, b]$ with $| x_1 - x_2 | < \delta$,

$$| f(x_1) - f(x_2) | < \varepsilon.$$

Hence $f$ is uniformly continuous on $[a, b]$.

## Theorem

*Let K be a compact set. If a function $f : K \to R$ is continuous then $f$ is uniformly continuous.*

***Proof.** Let $\varepsilon > 0$ be given.

Since $f$ is continuous at a point $\xi \in K$, there exists a $\delta(\xi) > 0$ ($\delta(\xi)$ depending upon $\varepsilon$ and $\xi$) such that $x \in K$ and $| x - \xi | < \delta(\xi)$

$$\Rightarrow \quad | f(x) - f(\xi) | < \frac{\varepsilon}{2}.$$

Now, the open sets $\left(\xi - \frac{\delta(\xi)}{2}, \xi + \frac{\delta(\xi)}{2}\right)$, $\xi \in K$, form an open covering of K.

Since K is compact, by Heine-Borel Theorem, there exist finitely many points $\xi_1, \xi_2, ......, \xi_n$ such that the sets $\left(\xi_i - \frac{\delta(\xi_i)}{2}\, \xi_i + \frac{\delta(\xi_i)}{2}\right)$, $1 \le i \le n$, form an open covering of K.

Let $\quad \delta = \min\limits_{1 \le i \le n} \delta(\xi_i) > 0$

Let $x, y \in K$ such that $| x - y | < \frac{\delta}{2}$ ...(1)

Let $\quad x \in \left(\xi_i - \frac{\delta(\xi_i)}{2}, \xi_i + \frac{\delta(\xi_i)}{2}\right)$ ...(2)

Now,
$$\begin{aligned} | \xi_i - y | &= | \xi_i - x + x - y | \\ &\le | \xi_i - x | + | x - y | \\ &< \frac{\delta(\xi_i)}{2} + \frac{\delta}{2} \qquad \text{[By (1) and (2)]} \\ &< \delta(\xi_i) \end{aligned}$$

$\Rightarrow \quad | f(\xi_i) - f(y) | < \frac{\varepsilon}{2}$ ...(3)

Since $\quad | x - \xi_i | < \frac{\delta(\xi_i)}{2}$, [By (2)]

*Not included in K.U. syllabus.

$\therefore \qquad | f(x) - f(\xi_i) | < \dfrac{\varepsilon}{2}$ ...(4)

Now, $\qquad | f(x) - f(y) | = | f(x) - f(\xi_i) + f(\xi_i) - f(y) |$

$$\leq | f(x) - f(\xi_i) | + | f(\xi_i) - f(y) |$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{[By (3) and (4)]}$$

Hence, $f$ is uniformly continuous on K.

**Cor.** *A function which is continuous on a closed interval [a, b], in uniformly continuous on that interval.*

## SOLVED EXAMPLES

**Example 1.** *Show that the function*

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

*is discontinuous at x = 0 and the discontinuity is of second kind.*

**Sol.** We shall show that $\lim_{x \to 0} f(x)$ does not exist.

Let, if possible, $\lim_{x \to 0} f(x)$ exist and be $l$.

Take $\varepsilon = \frac{1}{2}$. Then $\exists\, \delta > 0$, such that

$$\left| \sin\frac{1}{x} - l \right| < \frac{1}{2}, \text{ whenever } 0 < | x | < \delta$$

Let $0 < | x_1 | < \delta$ and $0 < | x_2 | < \delta$.

Then, $$\left| \sin\frac{1}{x_1} - \sin\frac{1}{x_2} \right| = \left| \left(\sin\frac{1}{x_1} - l\right) - \left(\sin\frac{1}{x_2} - l\right) \right|$$

$$\leq \left| \sin\frac{1}{x_1} - l \right| + \left| \sin\frac{1}{x_2} - l \right|$$

$$< \frac{1}{2} + \frac{1}{2} = 1.$$

Taking $x_1 = \dfrac{1}{2n\pi + \pi/2}, \quad x_2 = \dfrac{1}{2n\pi - \pi/2}$

with $n$ so large that $| x_1 |$ and $| x_2 |$ are both $< \delta$. (This is possible by Archimedean property of reals), we have

$$\left| \sin\left(2n\pi + \frac{\pi}{2}\right) - \sin\left(2n\pi - \frac{\pi}{2}\right) \right| < 1$$

*i.e.*, $\qquad 2 < 1 \qquad \left[\because \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1 \text{ and } \sin\left(2n\pi - \frac{\pi}{2}\right) = -1\right]$

This contradiction implies that $\lim_{x \to 0} \sin\dfrac{1}{x}$ does not exist.

**Example 2.** *Show that the function*

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

NOTES

*is continuous for every real x.*

**Sol.** Let $x \in R$

**Case** (*i*) $x = 0$.

Let $\varepsilon > 0$ be given. Then, for $\delta = \varepsilon$, we have

Now, $$| f(x) - f(0) | = \left| x \sin \frac{1}{x} - 0 \right| = \left| x \sin \frac{1}{x} \right|$$

$$= | x | \left| \sin \frac{1}{x} \right| \leq | x | < \varepsilon,$$

if $| x | < \varepsilon$ $\left[ \because \left| \sin \frac{1}{x} \right| < 1 \right]$

Taking $\delta = \varepsilon$, $| f(x) - f(0) | < \varepsilon$, whenever $| x | < \delta$

Thus, $f$ is continuous at $x = 0$.

**Case** (*ii*) $x \neq 0$.

Then, $1/x$ is continuous and $\sin 1/x$ is continuous. Since product of two continuous functions is continuous, it follows that $x \sin 1/x$ is continuous at $x$.

**Example 3.** *Dirichlet's Function. Show that the function*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

*is discontinuous for every real x.*

**Sol. Case** (*i*) *x is a rational number.*

Since in any interval there lie infinity many rationals as well as irrationals, for each $n \in N$, there exists an irrational number $x_n$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \quad \Rightarrow \quad | x_n - x | < \frac{1}{n}$$

$\Rightarrow$ the sequence $< x_n >$ is convergent to $x$.

But, $f(x_n) = -1 \quad \forall n$, and $f(x) = 1$

$\therefore$ $< f(x_n) >$ is convergent to $-1 \neq f$

$\therefore$ $f$ is discontinuous at $x$, any rational number.

**Case** (*ii*) *x is any irrational number.*

Since in any interval, there lie infinitely many rationals as well as irrationals, for each $n \in N$, there exists a rational number $x_n$ (say) such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \quad \Rightarrow \quad | x_n - x | < \frac{1}{n}$$

$\Rightarrow$ the sequence $< x_n >$ is convergent to $x$.

But, $f(x_n) = 1 \quad \forall n$, and $f(x) = -1$

$\therefore$ $< f(x_n) >$ is convergent to $1 \neq f(x)$

$\therefore$ $f$ is discontinuous at $x$, any irrational number.

**Example 4.** *Show that the function f defined by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$

*continuous only at x = 0.*

**Sol. Case** (*i*) *x is a non-zero rational number.*

For each $n \in N$, there exists an irrational number $x_n$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \quad \Rightarrow \quad | x_n - x | < \frac{1}{n}$$

NOTES

$\Rightarrow$ the sequence $< x_n >$ is convergent to $x$.

But, $f(x_n) = -x_n \quad \forall\, n \in N$ and $f(x) = x$

$\therefore$ $< f(x_n) >$ is convergent to $-x \neq f(x)$

$\therefore$ $f$ is discontinuous at $x$, any rational number.

**Case** (*ii*) *x is an irrational number.*

Proof is similar to case (*i*).

**Case** (*iii*) $x = 0$

Let $\varepsilon > 0$ be given.

Now, for a rational number $x$,

$$| f(x) - f(0) | = | x | < \varepsilon \quad \text{for} \quad | x | < \varepsilon$$

and for an irrational number $x$,

$$| f(x) - f(0) | = | -x | = | x | < \varepsilon \quad \text{for} \quad | x | < \varepsilon$$

$\therefore$ for any $\varepsilon > 0$, $\exists\, \delta\ (= \varepsilon) > 0$ such that

$$| f(x) - f(0) | < \varepsilon, \text{ whenever } | x - 0 | < \delta$$

$\Rightarrow$ $f$ is continuous at $x = 0$.

**Example 5.** *Show that the function f defined by*

$$f(x) = 2x^2 - 3x + 5$$

*is uniformly continuous on [– 2, 2].*

**Sol.** Let $\varepsilon > 0$ be given

Let $x_1$, $x_2$ be any two points of [– 2, 2]

Then, $| f(x_1) - f(x_2) | = | 2(x_1^2 - x_2^2) - 3(x_1 - x_2) | = | (x_1 - x_2)(2(x_1 + x_2) - 3) |$

$$= | x_1 - x_2 |\ | 2(x_1 + x_2) - 3 |$$

$$\leq | x_1 - x_2 |\ | 2( | x_1 | + | x_2 | ) + 3 |$$

$$\leq | x_1 - x_2 |\ [2(2 + 2) + 3] \qquad (\because\ | x | \leq 2)$$

$$= 11 \,.\, | x_1 - x_2 | < \varepsilon \text{ for } | x_1 - x_2 | < \frac{\varepsilon}{11}$$

$\therefore$ for any $\varepsilon > 0$, $\exists\, \delta \left(= \frac{\varepsilon}{11}\right)$ such that

$$| f(x_1) - f(x_2) | < \varepsilon \text{ for } | x_1 - x_2 | < \delta.$$

$\Rightarrow$ $f$ is uniformly continuous on [– 2, 2].

**Example 6.** *Prove that the function f defined by*

$$f(x) = \sin \frac{1}{x}\ ,\ x \in R^+$$

*is continuous but not uniformly continuous on $R^+$.*

**Sol.** For $x \in R^+$, $\frac{1}{x}$ is continuous and $\sin x$ is continuous for each $x$.

$\therefore$ $\sin \frac{1}{x}$ is continuous for $x \in R^+$.

Now for any $\delta > 0$, $\exists\, m \in \mathbb{N}$ such that $\dfrac{1}{2m\left(m\pi + \dfrac{\pi}{2}\right)} < \delta$

NOTES

Take $x_1 = \dfrac{1}{m\pi + \dfrac{\pi}{2}}$, $x_2 = \dfrac{1}{m\pi}$

Then $x_1, x_2 \in \mathbb{R}^+$

$$|x_1 - x_2| = \left| \frac{1}{m\pi + \frac{\pi}{2}} - \frac{1}{m\pi} \right| = \frac{1}{2m\left(m\pi + \frac{\pi}{2}\right)} < \delta$$

but $$|f(x_1) - f(x_2)| = \left| \sin\left(m\pi + \frac{\pi}{2}\right) - \sin m\pi \right| = |\cos m\pi| = 1$$

$(\because \sin m\pi = 0)$

which is not less than each $\varepsilon > 0$.

Hence $f$ is not uniformly continuous.

**Example 7.** *Examine for continuity the function f defined by*

$$f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n},\ 0 \le x \le \frac{\pi}{2}$$

*at x = 1. Explain why the function f does not vanish anywhere on* $\left[0, \frac{\pi}{2}\right]$ *although* $f(0).\, f\left(\frac{\pi}{2}\right) < 0$.

**Sol.** Since $$\lim_{n \to \infty} x^n = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \\ \infty & \text{if } 1 < x \le \frac{\pi}{2} \end{cases}$$

$$\therefore \quad f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n} = \begin{cases} e^x & \text{if } 0 \le x < 1 \\ \dfrac{e - \sin 1}{2} & \text{if } x = 1 \\ -\sin x & \text{if } 1 < x \le \frac{\pi}{2} \end{cases}$$

$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} e^x = e$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-\sin x) = -\sin 1$$

$$\therefore \quad \lim_{x \to 1^-} f(x) \ne \lim_{x \to 1^+} f(x)$$

$\Rightarrow$ $\lim_{x \to 1} f(x)$ does not exist

$\Rightarrow$ $f(x)$ is discontinuous at $x = 1$

Now $f(0) = e^0 = 1$.

Also $f\left(\dfrac{\pi}{2}\right) = -\sin \dfrac{\pi}{2} = -1$.

$\therefore \qquad f(0)\, f\left(\frac{\pi}{2}\right) = -1 < 0.$

NOTES

Since 1, the point of discontinuity of $f$ belongs to $\left[0, \frac{\pi}{2}\right]$, $f$ is not continuous on $\left[0, \frac{\pi}{2}\right]$. Intermediate value theorem is not applicable.

**Example 8.** *Prove that the function f defined by* $f(x) = x^2$, $x \in R$ *is uniformly continuous on every finite closed interval but is not uniformly continuous on R.*

**Sol.** Let $[a, b]$ be a finite closed interval.

Let $\varepsilon > 0$ be given.

Let $c = \max.\{|a|, |b|\} > 0$.

Let $x_1, x_2 \in [a, b]$ be any two numbers.

$\therefore \qquad |x_1| \le c, \; |x_2| \le c \qquad \ldots(1)$

Now, $\quad |f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 + x_2|\,|x_1 - x_2|$

$\le (|x_1| + |x_2|)\,|x_1 - x_2| \le 2c\,|x_1 - x_2| \qquad$ [By (1)]

$\therefore \qquad |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \frac{\varepsilon}{2c}$.

*i.e.*, $\qquad |f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$

where $\delta = \frac{\varepsilon}{2c} > 0$.

Thus $|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$, $\forall\, x_1, x_2 \in [a, b]$

$\therefore$ $f$ is uniformly continuous on $[a, b]$.

*To show that f is not uniformly continuous on R*

To show this, we have to show that for any given $\varepsilon > 0$ and any $\delta > 0$, we can find $x_1, x_2 \in$ R such that

$$|f(x_1) - f(x_2)| \ge \varepsilon \text{ but } |x_1 - x_2| < \delta.$$

Define a sequence $< a_n >$ such that

$$a_n = \sqrt{n + 2\varepsilon} - \sqrt{n} = \frac{2\varepsilon}{\sqrt{n + 2\varepsilon} + \sqrt{n}}$$

$\therefore$ as $n \to \infty$, $a_n \to 0$.

$\therefore$ for given $\delta > 0$, there exists a positive integer $m$ such that

$$|a_n - 0| < \delta \text{ whenever } n \ge m.$$

$$\Rightarrow \qquad |\sqrt{n + 2\varepsilon} - \sqrt{n}| < \delta \text{ whenever } n \ge m.$$

In particular, taking $n = m$,

$$|\sqrt{m + 2\varepsilon} - \sqrt{m}| < \delta \qquad \ldots(2)$$

Taking $x_1 = \sqrt{m + 2\varepsilon}$, and $x_2 = \sqrt{m}$ so that from (2)

$$|x_1 - x_2| < \delta \text{ but } |f(x_1) - f(x_2)| = |m + 2\varepsilon - m| = 2\varepsilon > \varepsilon.$$

Thus, we have found $x_1, x_2 \in$ R such that $|x_1 - x_2| < \delta$ but

$$|f(x_1) - f(x_2)| > \varepsilon.$$

Hence, $f$ is not uniformly continuous.

NOTES

## EXERCISE

1. Give one example of each of the following :
   (*i*) A function on [0, 1] which is continuous everywhere except the end points.
   (*ii*) a function on [1, 2] which is continuous everywhere except at 3/2.
   (*iii*) a function continuous on (0, 1) but is not bounded.
   (*iv*) a function which is bounded on (0, 1) but is not continuous.
   (*v*) a function $f$ continuous no where but $|f|$ continuous everywhere.
2. Examine the continuity of the following functions at the indicated point :

   (*i*) $f(x) = \dfrac{x^3 - 8}{x - 2}$, at $x = 2$

   (*ii*) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \neq 2 \\ 5, & \text{if } x = 2 \end{cases}$ at $x = 2$

   (*iii*) $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2 \end{cases}$ at $x = 2$

   (*iv*) $f(x) = \begin{cases} \dfrac{1}{x - 2}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases}$ at $x = 2$

   (*v*) $f(x) = \begin{cases} e^{-\frac{1}{(x-2)^2}}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases}$ at $x = 2$

   (*vi*) $f(x) = \begin{cases} x \cos \dfrac{1}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x \in$ R

   (*vii*) $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ at $x \in$ R

   (*viii*) $f(x) = \dfrac{|x|}{x}$ at $x = 0$

   (*ix*) $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ at $x = 0$

   (*x*) $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$ at $x = 0$

   (*xi*) $f(x) = [x]$ on $R^+$ at $x = 1, 2, 3, \ldots\ldots$

   (*xii*) $f(x) = \begin{cases} e^{\frac{1}{x}}; & x \neq 0 \\ 0; & x = 0 \end{cases}$

   Also, discuss the type of discontinuity in each case.
3. Discuss the continuity at 1, 2, 3 of the following function :

$$f(x) = \begin{cases} 5x + 4, & x \leq 1 \\ x^2 + 7x + 1, & 1 < x \leq 2 \\ x + 3, & 2 < x < 3 \\ 5x + 2, & x \geq 3 \end{cases}$$

4. Obtain the points of discontinuity of the function defined on [0, 1] as follows :

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} - x & \text{if } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ \frac{2}{3} - x & \text{if } \frac{1}{2} < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

   Also, examine the kinds of discontinuities.
5. (*a*) Show that the function $f$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

   is discontinuous at every point and discontinuity is of second kind.

   (*b*) Show that $f(x) = \begin{cases} 4, & \text{if } x \text{ is rational} \\ -3, & \text{if } x \text{ is irrational} \end{cases}$ is discontinuous for each real $x$.

**6.** Show that the function $f$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$$

has a discontinuity of second kind at every real number except 0.

**7.** (*a*) Show that the function $[x]$, where $[x]$ denotes the largest integer $\leq x$ is discontinuous at each integral value and the discontinuities are of first kind.

(*b*) Show that the function $x - [x]$ is discontinuous at every integral value of $x$ and all discontinuities are of first kind from left.

**8.** (*a*) If a function $f$ satisfies the inequality $f(x) \leq x, \forall x$, then show that $f$ is continuous at $x = 0$.

(*b*) Let $f$ be a continuous function at $x = 0$ and $f(0) = 0$. If $g$ is another function such that $| g(x) | \leq | f(x) | \ \forall x$, then $g$ is continuous at $x = 0$.

**9.** Show that the function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.

**10.** Give an example of each of the following :

(*i*) a continuous bounded function on R may attain the l.u.b. but may not attain the g.l.b.

(*ii*) a continuous bounded function on R which attains the g.l.b. but not the l.u.b.

(*iii*) a function continuous on an open interval but may fail to be uniformly continuous.

**11.** If $f$ and $g$ are two continuous functions on $[a, b]$ such that $f(a) < g(a)$ and $f(b) > g(b)$, then show that there exists a real number $c \in (a, b)$ such that $f(c) = g(c)$.

[**Hint.** Apply Intermediate Value Theorem to the continuous function $f - g$ on $[a, b]$.]

**12.** If $f : (0, \infty) \to R$ is a function defined by $f(x) = \dfrac{1}{x}$, prove that $f$ is uniformly continuous on $[a, \infty)$ where $a > 0$. Show that $f$ is continuous but not uniformly on $(0, \infty)$.

**13.** Show that the function :

$$\phi(y) = \begin{cases} y, & \text{if } y \text{ is irrational} \\ -y, & \text{if } y \text{ is rational} \end{cases}$$

is continuous only at $y = 0$.

[**Hint.** Proceed on lines of Example 5.]

**14.** If a function $f$ is continuous and strictly monotonically decreasing in $[a, b]$, then $f^{-1}$ is also continuous in $[f(b), f(a)]$.

[See Th. 6.4.14]

**15.** If a function $f$ is continuous on R, then prove that the set $B = \{x \mid x \in R \text{ and } f(x) < 0\}$ is an open set.

**16.** (*a*) Prove that the function $f$, defined by $f(x) = x^2 - 3x + 5$ is uniformly continuous on $[-3, 4]$.

(*b*) Show that the function of defined by $f(x) = 2x^2 + 3x - 4$ is uniformly continuous on $[0, 2]$.

**17.** Show that the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

is discontinuous at $x = 0$. Write the type of discontinuity.

**18.** Let $f(x) = \begin{cases} 1 & \text{if } x \leq 3 \\ ax + b & \text{if } 3 < x < 5 \\ 7 & \text{if } x \geq 5 \end{cases}$

Determine the constants $a$ and $b$ so that $f$ may be continuous.

**19.** Prove that the function $f(x) = x^2$ is not uniformly continuous on $(-\infty, \infty)$.

[See second part of example 9]

**20.** If $f(x)$ and $g(x)$ are continuous functions on $[a, b]$ then $f(x) + g(x)$ is also continuous on $[a, b]$.

NOTES

# DIFFERENTIABILITY

**STRUCTURE**



## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Derivability or differentiability at a Point
- Left and Right Hand Derivatives at a Point
- Derivability on an Interval
- Relation between Continuity and Differentiability
- Points to Remember

## INTRODUCTION

In the previous chapters, we studied the functions, limits and continuity. In this chapter, we will use the concept of limit to introduce the idea of differentiability. It help us to study rates at which physical quantities change.

## DERIVABILITY OR DIFFERENTIABILITY AT A POINT

Let $f(x)$ be a real valued function and $a$ be any point in its domain. Then, $f(x)$ is said to have a derivative at $x = a$ if and only if $f(x)$ is defined in some neighbourhood of $a$ and

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists finitely.}$$

where $h$ be any small but arbitrary (positive or negative) number.

The value of this limit is called the derivative of $f(x)$ at $x = a$ and is denoted by $f'(a)$.

*i.e.,* $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$



Now, $f(x)$ is differentiable at $x = a$, if and only if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists finitely.

Also, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists if and only if $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$

or $$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{-h} \text{ and } \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \text{ both exist and are equal.}$$

## LEFT AND RIGHT HAND DERIVATIVES AT A POINT

If the function $f(x)$ involves modulus function, bracket function and/or is defined by more than one rule, then $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ may depends upon the sign of increment $h$ of $x$. In such cases, we calculate,

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} \text{ separately.}$$

These limits are called Left Hand Derivative of $f(x)$ at $a$ and Right Hand Derivative of $f(x)$ at $a$ and are denoted by $Lf'(a)$ and $Rf'(a)$ respectively.

$\therefore$ Left Hand Derivative : $$Lf'(a) = \lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h}$$

Right Hand Derivative : $$Rf'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

It may be noted that, $Lf'(a) \neq Rf'(a)$ implies that $f(x)$ is not differentiable at $x = a$.

## DERIVABILITY ON AN INTERVAL

A function $f(x)$ is said to be derivable (or differentiable) on an open interval $(a, b)$ if : $f(x)$ is derviable at every point in $(a, b)$.

A function $f(x)$ is said to be derivable (or differentiable) on a closed interval $[a, b]$ if :

(*i*) It is derivable in the open interval $(a, b)$.
(*ii*) It is right derivable at '$a$' and left derivable at '$b$'.

## RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY

**Theorem.** *Every differentiable function is continuous, but every continuous function may or may not be differentiable.*

NOTES

**Proof.** Let $f(x)$ be a differentiable function and let $a$ be any point in its domain.

Then, $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad ...(1)$$

Now, $$\lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \times h\right]$$

$$= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h}\right) . \lim_{h \to 0} h$$

$$= f'(a) \times 0 \quad \text{[By using (1)]}$$

$$= 0$$

$$\Rightarrow \lim_{h \to 0} [f(a+h) - f(a)] = 0$$

$$\Rightarrow \lim_{h \to 0} f(a+h) = f(a).$$

This shows that $f(x)$ is continuous at $x = a$. So, every differentiable function is always continuous.

Now, in order to show that the converse is not true *i.e.,* every continuous function need not be differentiable, let us consider the following example :

Let $f(x) = |x|$ at $x = 0$.

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} |-h| = \lim_{h \to 0} (h) = 0.$$

And, $$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h| = \lim_{h \to 0} (h) = 0$$

Also, $$f(0) = |0| = 0.$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0.$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Now, let us check the derivability of $f(x) = |x|$ at $x = 0$.

$\therefore$ L.H.D, $$\text{L}f'(a) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{+h} \quad \left[\because f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right]$$

$$= \lim_{h \to 0^-} \frac{f(+h) - f(0)}{+h} = \lim_{h \to 0^-} \frac{|+h| - 0}{+h} = \lim_{h \to 0} \left(\frac{-h}{+h}\right) = -1.$$

R.H.D., $$\text{R}f'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \quad \left[\because f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right]$$

$$= \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h| - 0}{h} = \lim_{h \to 0} \left(\frac{h}{h}\right) = 1.$$

$$\therefore \text{L}f'(0) \neq \text{R}f'(0).$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

Hence, $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$.

**Remark.** (*i*) Continuity is not a sufficient condition *i.e.,* every continuous function may not be differentiable.

(*ii*) Continuity is a necessary condition for differentiablility *i.e.,* if a function is not continuous at some point $a$, then, it cannot be differentiated at that point.

NOTES

## POINTS TO REMEMBER

(*i*) All constant polynomials are differentiable for all $x \in$ R.

(*ii*) All polynomial functions are differentiable for all $x \in$ R.

(*iii*) The exponential function $a^x$ $(a > 0)$ is differentiable for all $x \in$ R.

(*iv*) The logarithmic function is differentiable at each point of its domain.

(*v*) Trigonometric and Inverse Trigonometric functions are differentiable in their domains.

(*vi*) The composite function of two differentiable function is also differentiable.

(*vii*) The sum, difference, product and quotient of two differentiable function is also differentiable.

## SOLVED EXAMPLES

**Example 1.** *If f is differentiable at x = a, then prove that :*

$$\lim_{x \to a} \frac{xf(a) - af(x)}{x - a} = f(a) - af'(a).$$

**Solution.** Since $f$ is differentiable at $x = a$.

$$\therefore \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad ...(1)$$

Now, L.H.S. we have, $\lim_{x \to a} \left[\frac{xf(a) - af(x)}{x - a}\right]$

Put $\qquad x = a + h \;\Rightarrow\; h \to 0$ as $x \to a$

$$\Rightarrow \qquad \lim_{x \to a} \frac{xf(a) - af(x)}{x - a} = \lim_{h \to 0} \left[\frac{(a+h)\,f(a) - af(a+h)}{a + h - a}\right]$$

$$= \lim_{h \to 0} \left[\frac{af(a) + hf(a) - af(a+h)}{h}\right]$$

$$= \lim_{h \to 0} \left(\frac{hf(a) - a\,[f(a+h) - f(a)]}{h}\right)$$

$$= \lim_{h \to 0} \left[\frac{hf(a)}{h} - \frac{a[f(a+h) - f(a)]}{h}\right]$$

$$= f(a) - a \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= f(a) - af'(a). \qquad \text{[By using (1)]}$$

NOTES

**Example 2.** *Show that the function f(x) = $x^2$ for x ≤ 0 and f(x) = x for x > 0 is not derivable at x = 0.*

**Solution.** We have, $f(x) = \begin{cases} x^2 & ; x \le 0 \\ x & ; x > 0 \end{cases}$

$$\therefore \quad \text{L.H.D. } Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} \qquad [\because \ f(x) = x^2 \text{ for } x \le 0]$$

$$= \lim_{h \to 0^-} \left(\frac{(h)^2 - 0}{h}\right) = \lim_{h \to 0^-} (h) = 0$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \qquad [\because \ f(x) = x \text{ for } x > 0]$$

$$= \lim_{h \to 0^+} \left(\frac{h - 0}{h}\right) = \lim_{h \to 0^+} 1 = 1$$

$$\therefore \quad Lf'(0) \ne Rf'(0)$$

⇒ $f'(0)$ does not exist.

⇒ $f(x)$ is not derivable at $x = 0$.

**Example 7.** *If f(x) is differentiable at x = a, then evaluate :*

$$\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}.$$

**Solution.** Since $f(x)$ is differentiable at $x = a$,

$$\therefore \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \ldots(1)$$

We have, $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$

Put $x = a + h \ \Rightarrow \ h \to 0$ as $x \to a$

$$\therefore \quad \lim_{x \to a} \left(\frac{x^2 f(a) - a^2 f(x)}{x - a}\right) = \lim_{h \to 0} \left[\frac{(a+h)^2 f(a) - a^2 f(a+h)}{a + h - a}\right]$$

$$= \lim_{h \to 0} \left(\frac{(a^2 + 2ah + h^2) f(a) - a^2 f(a+h)}{h}\right)$$

$$= \lim_{h \to 0} \left[\frac{a^2 f(a) + 2ahf(a) + h^2 f(a) - a^2 f(a+h)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{hf(a)(h + 2a) - a^2 [f(a+h) - f(a)]}{h}\right]$$

$$= \lim_{h \to 0} \left[\left(\frac{hf(a)(h+2a)}{h}\right) - a^2 \left(\frac{f(a+h) - f(a)}{h}\right)\right]$$

$$= \lim_{h \to 0} f(a)(h + 2a) - a^2 \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h}\right)$$

$$= f(a)(0 + 2a) - a^2 f'(a) \qquad [\because \ \text{By using (1)}]$$

$$= 2af(a) - a^2 f'(a).$$

**Example 4.** *Prove that the function f(x) defined by :*

$$f(x) = \begin{cases} x^2 + 1 & ; x \le 1 \\ 2x & ; x > 1 \end{cases} \text{ is differentiable at } x = 1.$$

**Solution.** We have,

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$$f(x) = \begin{cases} x^2 + 1 & ; x \le 1 \\ 2x & ; x > 1 \end{cases}$$

$$\text{L.H.D. } Lf'(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \qquad [\because \quad f(x) = x^2 + 1 \text{ for } x \le 1]$$

$$= \lim_{h \to 0^-} \left[ \frac{[(1+h)^2 + 1] - ((1)^2 + 1)}{h} \right]$$

$$= \lim_{h \to 0^-} \left( \frac{1 + h^2 + 2h + 1 - 2}{h} \right)$$

$$= \lim_{h \to 0^-} \left( \frac{h(h+2)}{h} \right) = \lim_{h \to 0^-} (h + 2) = 2.$$

$$\text{R.H.D. } Rf'(1) = \lim_{h \to 0^+} \left( \frac{f(1+h) - f(1)}{h} \right) \qquad [\because \quad f(x) = 2x \text{ for } x > 1]$$

$$= \lim_{h \to 0^+} \left( \frac{2(1+h) - 2(1)}{h} \right) = \lim_{h \to 0^+} \left( \frac{2 + 2h - 2}{h} \right)$$

$$= \lim_{h \to 0^+} \left( \frac{2h}{h} \right) = \lim_{h \to 0} (2) = 2.$$

$\therefore \qquad Lf'(1) = Rf'(1)$

$\therefore \quad f(x)$ is differentiable at $x = 1$.

**Example 5.** *Show that the function* $f(x) = |x - 1| + |x - 2|$ *is not derivable at* $x = 2$.

**Solution.** We have, $f(x) = |x - 1| + |x - 2|$

$$\text{L.H.D. } Lf'(2) = \lim_{h \to 0^-} \left( \frac{f(2+h) - f(2)}{h} \right)$$

$$= \lim_{h \to 0^-} \left( \frac{(|2 + h - 1| + |2 + h - 2|) - (|2 - 1| + |2 - 2|)}{h} \right)$$

$$= \lim_{h \to 0^-} \left( \frac{|1 + h| + |h| - |1| - 0}{h} \right)$$

$$= \lim_{h \to 0^-} \left( \frac{1 + h + (-h) - 1}{h} \right) \qquad [\because \quad |h| = -h \text{ for } h < 0]$$

$$= \lim_{h \to 0^-} \left( \frac{0}{h} \right) = 0.$$

$$\text{R.H.D. } Rf'(2) = \lim_{h \to 0^+} \left( \frac{f(2+h) - f(2)}{h} \right)$$

$$= \lim_{h \to 0^+} \left( \frac{(|2 + h - 1| + |2 + h - 2|) - (|2 - 1| + |2 - 2|)}{h} \right)$$

$$= \lim_{h \to 0^+} \left( \frac{|1 + h| + |h| - |1| - 0}{h} \right)$$

$$= \lim_{h \to 0^+} \left( \frac{1 + h + h - 1}{h} \right) = \lim_{h \to 0^+} \left( \frac{2h}{h} \right) = \lim_{h \to 0^+} (2) = 2.$$

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$\therefore \quad Lf'(2) \neq Rf'(2)$

$\Rightarrow \quad f(x)$ is not differentiable at $x = 2$.

**Example 6.** *Discuss the differentiability of the function :*

$$f(x) = \begin{cases} x e^{-\left( \frac{1}{|x|} + \frac{1}{x} \right)} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \text{ at } x = 0.$$

**Solution.** We have,

$$f(x) = \begin{cases} x e^{-\left( \frac{1}{|x|} + \frac{1}{x} \right)} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$$\therefore \quad \text{L.H.D. } Lf'(0) = \lim_{h \to 0^-} \left[ \frac{f(0+h) - f(0)}{h} \right]$$

$$= \lim_{h \to 0^-} \left[ \frac{(0+h)\, e^{-\left( \frac{1}{|0+h|} + \frac{1}{0+h} \right)}}{h} \right] = \lim_{h \to 0^-} \left[ \frac{h e^{-\left( \frac{1}{|h|} + \frac{1}{h} \right)}}{h} \right]$$

$$= \lim_{h \to 0^-} \left[ e^{-\left( \frac{1}{|h|} + \frac{1}{h} \right)} \right]$$

$$= \lim_{h \to 0^-} e^{-\left( \frac{1}{-h} + \frac{1}{h} \right)} = \lim_{h \to 0} e^0 = 1. \qquad [\because \ |h| = -h \text{ for } h < 0]$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \left( \frac{f(0+h) - f(0)}{h} \right)$$

$$= \lim_{h \to 0^+} \left[ \frac{(0+h)\, e^{-\left( \frac{1}{|0+h|} + \frac{1}{(0+h)} \right)} - 0}{h} \right] = \lim_{h \to 0^+} \left[ \frac{h\, e^{-\left( \frac{1}{|h|} + \frac{1}{h} \right)}}{h} \right]$$

$$= \lim_{h \to 0^+} \left[ e^{-\left( \frac{1}{|h|} + \frac{1}{h} \right)} \right] = \lim_{h \to 0^+} \left[ e^{-\left( \frac{1}{h} + \frac{1}{h} \right)} \right]$$

$$[\because \ |h| = h \text{ for } h > 0]$$

$$= \lim_{h \to 0^+} e^{-2/h} = \lim_{h \to 0^+} \frac{1}{e^{2/h}}$$

$$= \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$\therefore \quad Lf'(0) \neq Rf'(0)$

$\Rightarrow \quad f'(0)$ does not exist.

$\Rightarrow \quad f(x)$ is not differentiable at $x = 0$.

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**Example 7.** *Show that the function* $f(x) = \begin{cases} ax^2 + 1 & ; x \geq 1 \\ x + a & ; x < 1 \end{cases}$ *is continuous at* $x = 1$. *For what value of a, the function is differentiable at* $x = 1$.

**Solution.** We have,

$$f(x) = \begin{cases} ax^2 + 1 & ; x \geq 1 \\ x + a & ; x < 1 \end{cases}$$

**Continuity at x = 1 :**

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + a) = (1 + a)$

And, $\quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (ax^2 + 1) = (a + 1)$

Also, $\quad f(1) = a(1)^2 + 1 = (a + 1)$.

$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) = (a + 1)$

$\therefore \quad f(x)$ is continuous at $x = 1$ for all values of $a$.

**Derivability at x = 1 :**

$$\therefore \quad \text{L.H.D. } Lf'(1) = \lim_{h \to 0^-} \left[ \frac{f(1+h) - f(1)}{h} \right]$$

$$= \lim_{h \to 0^-} \left[ \frac{(1 + h + a) - (a + 1)}{h} \right] \qquad [\because \; f(x) = x + a \text{ for } x < 1]$$

$$= \lim_{h \to 0^-} \left( \frac{1 + h + a - a - 1}{h} \right) = \lim_{h \to 0^-} \left( \frac{h}{h} \right) = 1.$$

$$\text{R.H.D. } Rf'(1) = \lim_{h \to 0^+} \left[ \frac{f(1+h) - f(1)}{h} \right]$$

$$= \lim_{h \to 0^+} \left[ \frac{a(1+h)^2 + 1 - (a + 1)}{h} \right] \qquad [\because \; f(x) = ax^2 + 1 \text{ for } x \geq 1]$$

$$= \lim_{h \to 0^+} \left[ \frac{a(1 + h^2 + 2h) + 1 - a - 1}{h} \right]$$

$$= \lim_{h \to 0^+} \left[ \frac{a + ah^2 + 2ah - a}{h} \right] = \lim_{h \to 0^+} \left( \frac{ah^2 + 2ah}{h} \right)$$

$$= \lim_{h \to 0^+} \left[ \frac{h(ah + 2a)}{h} \right] = \lim_{h \to 0^+} (ah + 2a) = 2a$$

For differentiability of $f(x)$ at $x = 1$, we must have,

$$Lf'(1) = Rf'(1) \quad \Rightarrow \quad 1 = 2a \quad \Rightarrow \quad a = \frac{1}{2}$$

$\therefore \quad f(x)$ is differentiable at $x = 1$ for $a = \frac{1}{2}$.

**Example 8.** *For what choices of a, b and c, if any, does the function :*

$$f(x) = \begin{cases} ax^2 + bx + c & ; 0 \leq x \leq 1 \\ bx - c & ; 1 < x \leq 2 \\ c & ; x > 2 \end{cases}$$

*is differentiable at* $x = 1$ *and* $x = 2$.

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**Solution.** We have,

$$f(x) = \begin{cases} ax^2 + bx + c & ; 0 \le x \le 1 \\ bx - c & ; 1 < x \le 2 \\ c & ; x > 2 \end{cases}$$

**Differentiability at x = 1 :**

$\therefore$ L.H.D. $Lf'(1) = \lim_{h \to 0^-} \left[\frac{f(1+h) - f(1)}{h}\right]$ [$\because$ $f(x) = ax^2 + bx + c$ for $0 \le x \le 1$]

$$= \lim_{h \to 0^-} \left[\frac{\{a(1+h)^2 + b(1+h) + c\} - (a \,.\, 1^2 + b \,.\, 1 + c)}{h}\right]$$

$$= \lim_{h \to 0^-} \left(\frac{a(1+h^2+2h) + b + bh + c - a - b - c}{h}\right)$$

$$= \lim_{h \to 0^-} \left(\frac{a + ah^2 + 2ah + bh - a}{h}\right) = \lim_{h \to 0^-} \left[\frac{h(ah + 2a + b)}{h}\right]$$

$$= \lim_{h \to 0^-} (ah + 2a + b)$$

$\Rightarrow$ $Lf'(1) = 2a + b$ ...(1)

R.H.D. $Rf'(1) = \lim_{h \to 0^+} \left[\frac{f(1+h) - f(1)}{h}\right]$

$$= \lim_{h \to 0^+} \left[\frac{(b(1+h) - c) - (a.1^2 + b.1 + c)}{h}\right]$$

[$\because$ $f(x) = bx - c$ for $1 < x \le 2$]

$$= \lim_{h \to 0^+} \left[\frac{b + bh - c - a - b - c}{h}\right]$$

$$= \lim_{h \to 0^+} \left(\frac{bh - a - 2c}{h}\right) = \lim_{h \to 0^+} \left(\frac{bh}{h} - \frac{a + 2c}{h}\right)$$

$$Rf'(1) = \lim_{h \to 0^+} \left(b - \frac{a + 2c}{h}\right)$$

Since, $f'(1)$ exists, therefore we must have

$a + 2c = 0$

And, $b = 2a + b \Rightarrow 2a = 0 \Rightarrow a = 0$

Also, $a + 2c = 0 \Rightarrow c = 0$ [$\because$ $a = 0$]

$\therefore$ $a = 0$ and $c = 0$.

**Differentiability at x = 2 :**

$\therefore$ L.H.D., $Lf'(2) = \lim_{h \to 0^-} \left[\frac{f(2+h) - f(2)}{h}\right]$

$$= \lim_{h \to 0^-} \left[\frac{(b(2+h) - c) - (b.2 - c)}{h}\right]$$

[$\because$ $f(x) = bx - c$ for $1 < x \le 2$]

$$= \lim_{h \to 0^-} \left[\frac{2b + bh - c - 2b + c}{h}\right] = \lim_{h \to 0^-} \left(\frac{bh}{h}\right) = \lim_{h \to 0^-} (b) = b.$$

$$\text{R.H.D. } Rf'(2) = \lim_{h \to 0^+} \left[\frac{f(2+h) - f(2)}{h}\right]$$

$$= \lim_{h \to 0^+} \left[\frac{c - (b.2 - c)}{h}\right] = \lim_{h \to 0^+} \left[\frac{c - 2b + c}{h}\right]$$

[∵ $f(x) = c$ for $x > 2$]

$$= \lim_{h \to 0^+} \left(\frac{2c - 2b}{h}\right)$$

Since, $f'(2)$ exists, therefore we must have

$2c - 2b = 0 \Rightarrow 2c = 2b \Rightarrow b = 0$ [∵ $c = 0$]

∴ $a = 0, b = 0$ and $c = 0$.

**Example 9.** *Show that the function $f(x) = x\,|x|$ is differentiable at $x = 0$.*

**Solution.** We have, $f(x) = x\ |x|$

$$\text{L.H.D. } Lf'(0) = \lim_{h \to 0^-} \left[\frac{f(0+h) - f(0)}{h}\right] = \lim_{h \to 0^-} \left[\frac{(0+h)\,|0+h| - 0}{h}\right]$$

$$= \lim_{h \to 0^-} \left[\frac{h\,|h| - 0}{h}\right]$$

$$= \lim_{h \to 0^-} |h|$$ [∵ $|h| = -h$ for $h < 0$]

$$\Rightarrow \lim_{h \to 0^-} (-h) = 0$$

$$\text{R.H.D. } Rf'(0) = \lim_{h \to 0^+} \left[\frac{f(0+h) - f(0)}{h}\right]$$

$$= \lim_{h \to 0^+} \left[\frac{(0+h)\,|0+h| - 0}{h}\right]$$

$$= \lim_{h \to 0^+} \left[\frac{h\,|h|}{h}\right]$$ [∵ $|h| = h$ for $h > 0$]

$$= \lim_{h \to 0^+} |h| = \lim_{h \to 0^+} h = 0$$

∴ $Lf'(0) = Rf'(0)$

⇒ $f'(1)$ exists.

⇒ $f(x)$ is differentiable at $x = 0$.

## EXERCISE

1. Examine the derivability of the function :

$$f(x) = \begin{cases} 3 - 2x & ; x < 4 \\ 2x - 7 & ; x \geq 4 \end{cases} \text{ at } x = 4.$$

2. Find the value of $p$, if the function :

$$f(x) = \begin{cases} px^2 + 1 & ; x \geq 1 \\ x + p & ; x < 1 \end{cases}$$ is differentiable at $x = 1$.

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3. Show that $f(x) = |x - 5|$ is continuous but not differentiable at $x = 5$.

4. Let $$f(x) = \begin{cases} 2 + x & ; x \geq 0 \\ 2 - x & ; x < 0 \end{cases}.$$

Show that $f(x)$ is not derivable at $x = 0$.

5. Show that $$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$ is derivable at $x = 0$ and $f'(0) = 0$.

6. If $f(x)$ is derivable at $x = a$, then,

Prove that : $$\lim_{x \to a} \frac{xf(a) - af(x)}{x - a} = f(a) - af'(a).$$

7. Show that $f(x) = |x - 2|$ is continuous but not derivable at $x = 2$.

8. Find the values of $a$ and $b$ so that the function :

$$f(x) = \begin{cases} x^2 + 3x + a & ; x \leq 1 \\ bx + 2 & ; x > 1 \end{cases}$$ is differentiable at each $x \in$ R.

## Answers

1. Not derivable
2. $p = \dfrac{1}{2}$
8. $a = 3$ and $b = 5$.

# SUCCESSIVE DIFFERENTIATION

NOTES

**STRUCTURE**



## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Derivatives of $n$th Order of Some Standard Functions of $x$
- Use of Partial Fractions
- Leibnitz's Theorem
- Determination of the Value of the $n$th Derivative of a Function for $x = 0$

## INTRODUCTION

If $y$ be a function of $x$, its derivative $\frac{dy}{dx}$ is itself a function of $x$. In general, we assume that it also possesses a derivative if it is differentiated further. The derivative $\frac{dy}{dx}$ is called the first differential co-efficient or first derivative of $y$ w.r.t. $x$. The differential co-efficient of $\frac{dy}{dx}$, *i.e.*, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second differential co-efficient or **second derivative** of $y$ w.r.t. $x$, which we denote as $\frac{d^2y}{dx^2}$ [read as *dee two y* over *dee x squared*]. In like manner the third differential co-efficient or third derivative of $y$ w.r.t. $x$ means the differential co-efficient of $\frac{d^2y}{dx^2}$, *i.e.*, $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and is represented by $\frac{d^3y}{dx^3}$ and so on. In general, the $n$th differential co-efficient of $y$ w.r.t. $x$ is denoted by $\frac{d^ny}{dx^n}$. This process of finding the differential co-efficient of a function is called **Successive Differentiation.**

Thus, if $y = f(x)$, the successive differential co-efficients of $f(x)$ are

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots\ldots\ldots, \frac{d^ny}{dx^n}.$$

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These are also denoted by :

(*i*) $y_1, y_2, y_3, \ldots\ldots, y_n$. (*ii*) $y', y'', y''', \ldots\ldots, y^n$.

(*iii*) $Dy, D^2y, D^3y, \ldots\ldots D^ny$. (*iv*) $f(x), f'(x), f''(x), \ldots\ldots, f^n(x)$.

**Note 1.** The following formulae on trigonometry will be helpful in learning this chapter.

**1.** $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$ **2.** $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

**3.** $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$ **4.** $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

**5.** $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$ **6.** $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$

**7.** $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ **8.** $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

**Note. 2.** The following formulae on differentiation will be helpful in learning this chapter and subsequent chapters.

**1.** $\frac{d}{dx}(c)$ where $c$ is a constant.

**2.** (*a*) $\frac{d}{dx}(x^n) = nx^{n-1}$

(*b*) $\frac{d}{dx}(u^n)$ where $u$ is a function of $x = nu^{n-1}\frac{du}{dx}$

**3.** $\frac{d}{dx}(cu) = c\frac{du}{dx}$ where $c$ is a constant and $u$ is a function $x$.

**4.** $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ where $u$ and $v$ are functions of $x$.

**5.** $\frac{d}{dx} = \left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

**6.** (*a*) $\frac{d}{dx}(\sin x) = \cos x$ (*b*) $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$

**7.** (*a*) $\frac{d}{dx}(\cos x) = -\sin x$ (*b*) $\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$

**8.** (*a*) $\frac{d}{dx}(\tan x) = \sec^2 x$ (*b*) $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$

**9.** (*a*) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ (*b*) $\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u\frac{du}{dx}$

**10.** (*a*) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (*b*) $\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$

**11.** (*a*) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ (*b*) $\frac{d}{dx}(\operatorname{cosec} u) = -\operatorname{cosec} u \cot u\frac{du}{dx}$

**12.** (*a*) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$ (*b*) $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}$

**13.** (*a*) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$ (*b*) $\frac{d}{dx}\cos^{-1} u = \frac{-1}{\sqrt{1 - u^2}}\frac{du}{dx}$

**14.** (*a*) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$ (*b*) $\frac{d}{dx}\tan^{-1} u = \frac{1}{1 + u^2}\frac{du}{dx}$

**15.** (*a*) $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1 + x^2}$ (*b*) $\frac{d}{dx}\cot^{-1} u = \frac{-1}{1 + u^2}\frac{du}{dx}$

**16.** (*a*) $\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$ (*b*) $\frac{d}{dx}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$

**17.** (*a*) $\frac{d}{dx}\text{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$ (*b*) $\frac{d}{dx}\text{cosec}^{-1} u = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx}$

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**18.** (*a*) $\frac{d}{dx} e^x = e^x$ (*b*) $\frac{d}{dx} e^u = e^u \frac{du}{dx}$

**19.** (*a*) $\frac{d}{dx} a^x = a^x \log a$ (*b*) $\frac{d}{dx} a^u = a^u \log a \frac{du}{dx}$

**20.** (*a*) $\frac{d}{dx}\log x = \frac{1}{x}$ (*b*) $\frac{d}{dx}\log u = \frac{1}{u}\frac{du}{dx}$

**21.** (*a*) $\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e$ (*b*) $\frac{d}{dx}\log_a u = \frac{1}{u}\log_a e$

**22.** (*a*) $\sinh x = \frac{e^x - e^{-x}}{2}$ (*b*) $\cosh x = \frac{e^x + e^{-x}}{2}$

(*c*) $\cosh^2 x - \sinh^2 x = 1$

**23.** $\frac{d}{dx}\sinh x = \cosh x$

**24.** $\frac{d}{dx}\cosh x = \sinh x.$

## SOLVED EXAMPLES

**Example 1.** *Find* $\frac{d^3y}{dx^3}$, *when* $y = 4x^3 + 4x + 2$.

**Solution.** Here $y = 4x^3 + 4x + 2$

$$\therefore \quad \frac{dy}{dx} = 12x^2 + 4 \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}(12x^2+4) = 24x$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(24x) = 24.$$

**Example 2.** *If* $y = \frac{x}{\sqrt{1-x^2}}$, *find* $\frac{d^3y}{dx^3}$.

**Solution.** $y = \frac{x}{\sqrt{1-x^2}}$.

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{1-x^2}\,.\,1 - x\,.\,\frac{1}{2}(1-x^2)^{-1/2}\times(-2x)}{(1-x^2)}$$

$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{1-x^2} = \frac{1-x^2+x^2}{(1-x^2)^{3/2}} = (1-x^2)^{-3/2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{3}{2}(1-x^2)^{-5/2}\times(-2x) = 3x(1-x^2)^{-5/2}$$

and $$\therefore \quad \frac{d^3y}{dx^3} = 3\left[x\left(-\frac{5}{2}\right)(1-x^2)^{-7/2}\times(-2x) + (1-x^2)^{-5/2}\right]$$

$$= 15x^2(1-x^2)^{-7/2} + 3(1-x^2)^{-5/2}$$

$$= \frac{15x^2}{(1-x^2)^{7/2}} + \frac{3}{(1-x^2)^{5/2}} = \frac{15x^2 + 3(1-x^2)}{(1-x^2)^{7/2}}$$

**NOTES**

$$= \frac{3+12x^2}{(1-x^2)^{7/2}} = \frac{3\,(1+4x^2)}{(1-x^2)^{7/2}}.$$

**Example 3.** *If $y = a \cos (\log x) + b \sin (\log x)$, prove that $x^2y_2 + xy_1 + y = 0$.*

**Solution.** $y = a \cos (\log x) + b \sin (\log x)$ ...(1)

Differentiating w.r.t. $x$,

$$y_1 = -a \sin (\log x) \,.\, \frac{1}{x} + b \cos (\log x) \,.\, \frac{1}{x}$$

Multiplying every term by L.C.M. = $x$,

$$xy_1 = -a \sin (\log x) + b \cos (\log x)$$

Again differentiating both sides w.r.t. $x$

$$\frac{d}{dx}(xy_1) = -a \cos (\log x) \,.\, \frac{1}{x} - b \sin (\log x) \,.\, \frac{1}{x}$$

or $$x \,.\, y_2 + y_1 \,.\, 1 = -\frac{[a \cos (\log x) + b \sin (\log x)]}{x}$$

Cross-multiplying $x^2y_2 + xy_1 = -y$ [By (1)]

or $$x^2y_2 + xy_1 + y = 0.$$

**Example 4.** *If $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, prove that*

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}.$$

**Solution.** $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$

Differentiating w.r.t. $\theta$, we have

$$2p \,.\, \frac{dp}{d\theta} = a^2 \,.\, 2 \cos \theta \,(-\sin \theta) + b^2 \,.\, 2 \sin \theta \cos \theta$$

$$= (b^2 - a^2) \sin 2\theta \quad ...(1)$$

Differentiating again, we have

$$2p \,.\, \frac{d^2p}{d\theta^2} + 2\, \frac{dp}{d\theta} \cdot \frac{dp}{d\theta} = (b^2 - a^2) \times 2 \cos 2\theta$$

$\therefore$ $$p \,.\, \frac{d^2p}{d\theta^2} = (b^2 - a^2) \cos 2\theta - \left(\frac{dp}{d\theta}\right)^2$$

$$= (b^2 - a^2) \cos 2\theta - \frac{(b^2 - a^2)^2 \sin^2 2\theta}{4p^2} \quad \text{[From (1)]}$$

Adding $p^2$ to both sides, we have

$$p\frac{d^2p}{d\theta^2} + p^2 = (b^2 - a^2) \cos 2\theta - \frac{(b^2 - a^2)^2 \sin^2 2\theta}{4p^2} + p^2$$

or $$p\left(\frac{d^2p}{d\theta^2} + p\right) = \frac{4\,(b^2 - a^2) \cos 2\theta.\, p^2 - (b^2 - a^2)^2 \sin^2 2\theta + 4p^4}{4p^2}$$

Putting the value of $p^2$ in R.H.S.

$$= \frac{4(b^2 - a^2)(\cos^2\theta - \sin^2\theta)(a^2\cos^2\theta + b^2\sin^2\theta) - 4(b^2-a^2)^2 \sin^2\theta\cos^2\theta + 4(a^2\cos^2\theta + b^2\sin^2\theta)^2}{4p^2}$$

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or $$p^3\left(\frac{d^2p}{d\theta^2} + p\right) = a^2b^2\cos^4\theta + 2a^2b^2\sin^2\theta\cos^2\theta + a^2b^2\sin^4\theta$$

$$= a^2b^2(\cos^2\theta + \sin^2\theta)^2 = a^2b^2.$$

$$\therefore \quad \frac{d^2p}{d\theta^2} + p = \frac{a^2b^2}{p^3}.$$

**Example 5.** *If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$, prove that*

$$a\theta\frac{d^2y}{dx^2} = \sec^3\theta.$$

**Sol.** We have, $$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta) = a\theta\cos\theta \qquad ...(1)$$

and $$\frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta) = a\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Again, differentiating both sides w.r.t. *x*,

$$\frac{d^2y}{dx^2} = \sec^2\theta\ \frac{\mathbf{d\theta}}{\mathbf{dx}} = \sec^2\theta \cdot \frac{1}{a\theta\cos\theta}. \qquad \text{(By (1))}$$

$$= \frac{\sec^3\theta}{a\theta}$$

Hence $$a\theta\frac{d^2y}{dx^2} = \sec^3\theta.$$

## EXERCISE A

**1.** (*i*) If $y = x^3\log x$, prove that $y_4 = \frac{6}{x}$.

(*ii*) If $f(x) = \tan x$, find $f^{iv}(x)$, when $x = \frac{\pi}{4}$.

(*iii*) If $y = \frac{\log x}{x}$, prove that $\frac{d^2y}{dx^2} = \frac{2\log x - 3}{x^3}$.

(*iv*) If $f(x) = x^3\sin x$, find $f^{iv}(x)$.

**2.** If $y = A\sin nx + B\cos nx$, prove that $\frac{d^2y}{dx^2} + n^2y = 0$.

**3.** (*i*) If $y = (\sin^{-1}x)^2$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$.

[**Hint.** Differentiate w.r.t. $x$ ; cross-multiply and square, again differentiate.]

(*ii*) If $y = \tan^{-1}x$, show that $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$.

(*iii*) If $y = (\tan^{-1}x)^2$, prove that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$.

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**4.** (*i*) If $y = a\,e^{mx} + b\,e^{-mx}$, prove that $\frac{d^2y}{dx^2} - m^2y = 0$

(*ii*) If $y = e^{ax}\sin bx$, prove that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)\,y = 0$.

**5.** (*i*) If $y = \cos^{-1} x$, show that $y_2(1 - x^2) - xy_1 = 0$.

(*ii*) If $x = (a + bt)\,e^{-nt}$, prove that $\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + n^2x = 0$.

(*iii*) If $y = a\cosh\frac{x}{a}$, prove that $\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$.

**6.** If $y = e^{m\sin^{-1}x}$, show that $(1 - x^2)\,y_2 - xy_1 = m^2y$.

**7.** (*a*) If $y = A\,[x + \sqrt{1 + x^2}\,]^n$, prove that $(x^2 + 1)y_2 + xy_1 - n^2y = 0$.

(*b*) If $y = A\,(x + \sqrt{x^2 - 1})^n$, prove that $(x^2 - 1)\,y_2 + xy_1 - n^2y = 0$.

**8.** If $y = [\log(x + \sqrt{1 + x^2})]^2$, show that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$.

**9.** If $y = \frac{\sin^{-1}x}{\sqrt{1 - x^2}}$, show that

(*i*) $(1 - x^2)\,y_1 - xy = 1$ (*ii*) $(1 - x^2)\,y_2 - 3xy_1 - y = 0$.

**10.** (*a*) If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$.

(*b*) If $y = x + \cot x$, show that $\sin^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$.

(*c*) If $y = \tan^{-1}\left(\frac{\sqrt{1 + x^2} - 1}{x}\right) + \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$, show that $y_2 = -\frac{5x}{(1 + x^2)^2}$.

(*d*) If $y = \tan x + \sec x$, prove that $\frac{d^2y}{dx^2} = \frac{\cos x}{(1 - \sin x)^2}$.

**11.** (*i*) If $y = \frac{ax + b}{cx + d}$, show that $2y_1y_3 = 3y_2^2$.

(*ii*) If $y = \log(1 + \cos x)$, prove that $y_1y_2 + y_3 = 0$.

**[Hint for** (*i*) **and** (*ii*). Differentiate three times and put the values in L.H.S. and R.H.S.]

(*iii*) If $y = a\,e^x + b\,e^{2x} + c\,e^{3x}$, prove that $y_3 - 6y_2 + 11y_1 - 6y = 0$.

**12.** If $x = at^2$, $y = 2at$, find $\frac{d^2y}{dx^2}$.

**13.** (*i*) If $x = a\cos t$, $y = b\sin t$, find $\frac{d^2y}{dx^2}$.

(*ii*) If $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{2}$.

**14.** If $x = 2\cos t - \cos 2t$ and $y = 2\sin t - \sin 2t$, find the value of $\frac{d^2y}{dx^2}$, when $t = \frac{\pi}{2}$.

**15.** (*a*) If $x = \sin t$, $y = \sin pt$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$.

(*b*) If $= \log t$ and $y = \frac{1}{t}$ $(t > 0)$, prove that $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

**16.** If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, prove that $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$.

**17.** If $x^{1/2} + y^{1/2} = a^{1/2}$, find the value of $\dfrac{d^2y}{dx^2}$ for $x = a$.

**18.** If $ax^2 + 2hxy + by^2 = 1$, show that $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx + by)^3}$.

**19.** If $y^3 + x^3 - 3\,axy = 0$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^3xy}{(y^2 - ax)^3}$.

**20.** If $y = e^{-x}\cos x$, show that $\dfrac{d^4y}{dx^4} + 4y = 0$.

NOTES

## Answers

**1.** (*ii*) 80    (*iv*) $x^3 \sin x - 12x^2 \cos x - 36x \sin x + 24 \cos x$

**12.** $\dfrac{-1}{2at^3}$.    **13.** (*i*) $\dfrac{-b}{a^2}\,\text{cosec}^3\, t$    (*ii*) $\dfrac{-1}{a}$

**14.** $\dfrac{-3}{2}$    **17.** $\dfrac{1}{2a}$.

## Hints and Solutions

**4.** (*ii*)
$$y = e^{ax}\sin bx \qquad ...(1)$$
$\therefore$
$$\frac{dy}{dx} = b\,e^{ax}\cos bx + a\,e^{ax}\sin bx$$
or
$$\frac{dy}{dx} = b\,e^{ax}\cos bx + ay \qquad ...(2)\ \text{[by (1)]}$$
$\therefore$
$$\frac{d^2y}{dx^2} = -\,b^2 e^{ax}\sin bx + a\,b\,e^{ax}\cos bx + a\frac{dy}{dx}$$
$$= -\,b^2y + a\left[\frac{dy}{dx} - ay\right] + a\frac{dy}{dx}. \qquad \text{[by (1) and by (2)]}$$

**9.**
$$y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$$
$\therefore$
$$y_1 = \frac{\sqrt{1-x^2}\,\dfrac{d}{dx}\sin^{-1} x - \sin^{-1} x\,\dfrac{d}{dx}\sqrt{1-x^2}}{(1 - x^2)}$$
or
$$y_1 = \frac{\sqrt{1-x^2}\,\dfrac{1}{\sqrt{1-x^2}} - \sin^{-1} x\,.\,\dfrac{1}{2}(1-x^2)^{-1/2}(-2x)}{1 - x^2}$$
Cross-multiplying, $(1 - x^2)\,y_1 = 1 + x\dfrac{\sin^{-1} x}{\sqrt{1-x^2}} = 1 + xy$

**10.** (*a*)
$$y = \sin(\sin x) \qquad ...(1)$$
$\therefore$
$$\frac{dy}{dx} = \cos(\sin x)\,.\,\cos x \qquad ...(2)$$
$\therefore$
$$\frac{d^2y}{dx^2} = -\cos(\sin x)\sin x - \cos x.\,\sin(\sin x)\,.\,\cos x$$
$$= -\frac{\dfrac{dy}{dx}}{\cos x}\,.\,\sin x - \cos^2 x\,.\,y \qquad \text{[by (2) and by (1)]}$$

(*c*) Put $x = \tan\theta$

(*d*) $$\frac{dy}{dx} = \sec^2 x + \sec x \tan x = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1}{1-\sin x}.$$

**NOTES**

**14.** $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t - 2\cos 2t}{-2\sin t + 2\sin 2t}$$

$$= \frac{\cos t - \cos 2t}{\sin 2t - \sin t} = \frac{2\sin\frac{t+2t}{2}\,\sin\frac{2t-t}{2}}{2\cos\frac{2t+t}{2}\,\sin\frac{2t-t}{2}} \qquad | \text{ C} - \text{D Formulae}$$

$$= \frac{2\sin\frac{3t}{2}\sin\frac{t}{2}}{2\cos\frac{3t}{2}\sin\frac{t}{2}} = \tan\frac{3t}{2}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\tan\frac{3t}{2}\right) = \sec^2\frac{3t}{2}\,\frac{d}{dx}\left(\frac{3t}{2}\right)$$

$$= \sec^2\frac{3t}{2}\,.\,\frac{3}{2}\frac{dt}{dx} = \frac{3}{2}\sec^2\frac{3t}{2}\,\frac{1}{-2\sin t + 2\sin 2t}.$$

**15.** $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t} \quad \text{or} \quad \frac{dy}{dx} = \frac{p\sqrt{1-\sin^2 pt}}{\sqrt{1-\sin^2 t}} = \frac{p\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

Squaring both sides and cross-multiplying,

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2).$$

Again differentiate w.r.t. $x$.

**16.** $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(1)$$

$\therefore$ $$\frac{d}{dx}\left(\frac{x^2}{a^2}\right) + \frac{d}{dx}\left(\frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

or $$\frac{1}{a^2}\frac{d}{dx}(x^2) + \frac{1}{b^2}\frac{d}{dx}(y^2) = 0 \quad \text{or} \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$\therefore$ $$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2} \quad \text{or} \quad \frac{dy}{dx} = -\frac{2x}{a^2}\times\frac{b^2}{2y} = -\frac{b^2x}{a^2y} \qquad ...(2)$$

Again differentiating both sides w.r.t. $x$,

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right) = -\frac{b^2}{a^2}\,\frac{\left[y\,.\,1 - x\frac{dy}{dx}\right]}{y^2}$$

$$= \frac{-\frac{b^2}{a^2}\left[y + \frac{b^2x^2}{a^2y}\right]}{y^2} \qquad \text{[by (2)]}$$

$$= -\frac{b^2(a^2y^2 + b^2x^2)}{a^4y^3} = \frac{-b^2\,.\,a^2b^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)}{a^4y^3}$$

$$= -\frac{b^4}{a^2y^3}\times 1. \qquad \text{[by (1)]}$$

## DERIVATIVES OF THE *n*TH ORDER OF SOME STANDARD FUNCTIONS OF *x*

NOTES

**I. To find the nth differential of $x^m$.**

Let $y = x^m$

then $y_1 = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)x^{m-3}$

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$\therefore$ $y_n = [m(m-1)(m-2) \text{ .......... upto } n \text{ factors}] \times x^{m-n}$

$= m(m-1)(m-2) \text{ ........ } (m-n+1) \,.\, x^{m-n}$ where $n < m$.

[**Cor.** If $m$ be a positive integer, and if $n = m$

then $y_m = m(m-1)(m-2) \text{ ......... } (m-m+1)\, x^{m-m}$

$= m(m-1)(m-2) \text{ ......... } 3.2.1. = m\,!$

*i.e.*, $$\frac{d^m}{dx^m}(x^m) = m\,!$$

and $(m+1)$th, $(m+2)$th derivative ........etc., each will be = 0.

$\therefore$ $y_m = m\,!$ which is constant and $y_{(m+1)}$ which is the derivative of $y_m$ will be zero and so on.]

**II. The nth differential co-efficient of $(ax + b)^m$, where $n < m$.**

Let $y = (ax+b)^m$

then $y_1 = m(ax+b)^{m-1} \,.\, a$

$y_2 = m(m-1)(ax+b)^{m-2} \,.\, a^2$

$y_3 = m(m-1)(m-2)(ax+b)^{m-3} \,.\, a^3$

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$\therefore$ $y_n = m(m-1)(m-2)(m-3) \text{ ..... } (m-n+1) \times (ax+b)^{m-n} \,.\, a^n$

[**Cor.** If $n = m$, then

$y_m = m(m-1)(m-2) \text{ ..... } 3.2.1.\ (ax+b)^0\, a^m = m\,!\, a^m$]

**III. The nth differential co-efficient of $\frac{1}{ax+b}$ $\left[x \neq -\frac{b}{a}\right]$**

Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$

then $y_1 = (-1)(ax+b)^{-2} \,.\, a$

$y_2 = (-1)(-2)(ax+b)^{-3} \,.\, a^2 = (-1)^2 \,.\, 2\,!\,(ax+b)^{-3} \,.\, a^2$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \,.\, a^3$

$= (-1)^3 \,.\, 3\,!\,(ax+b)^{-4} \,.\, a^3$

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$\therefore$ $y_n = (-1)^n \,.\, n\,!\,(ax+b)^{-(n+1)} \,.\, a^n$

or $$y_n = \frac{(-1)^n\, n\,!\, a^n}{(ax+b)^{n+1}}.$$

NOTES

**IV. The nth differential co-efficient of log (ax + b)**

Let $y = \log (ax + b)$

then $y_1 = \dfrac{1}{ax+b}\, a = (ax + b)^{-1} . a$

$y_2 = (-1)(ax + b)^{-2} . a^2$

$y_3 = (-1)(-2)(ax + b)^{-3} . a^3 = (-1)^2\, 2\,!\,(ax + b)^{-3} . a^3$

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$\therefore$ $y_n = (-1)^{n-1}(n-1)\,!\,(ax + b)^{-n} . a^n$

or $$\mathbf{y_n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}}$$

$\left[\textbf{Cor.} \text{ If } a = 1; b = 0, y = \log x, \text{ then } y_n = \dfrac{(-1)^{n-1}(n-1)!}{x^n}.\right]$

**V. The nth differential co-efficient of $a^{mx}$**

Let $y = a^{mx}$

$\therefore$ $y_1 = m . a^{mx} (\log a)$

$y_2 = m^2 . a^{mx} (\log a)^2$

$y_3 = m^3 . a^{mx} (\log a)^3$

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$\therefore$ $\mathbf{y_n = m^n . a^{mx} (\log a)^n}$

[**Cor.** Put $a = e$, then $y = \mathbf{e^{mx}}$ and $\mathbf{y_n = m^n e^{mx}}$.]

**VI. The nth differential co-efficient of sin (ax + b)**

Let $y = \sin (ax + b)$

then $y_1 = a \cos (ax + b)$

$= a \sin\left(ax + b + \dfrac{\pi}{2}\right)$

$\because \sin\left(\dfrac{\pi}{2} + \theta\right) = \cos\theta$ [Note this step]

$y_2 = a^2 \cos\left(ax + b + \dfrac{\pi}{2}\right) = a^2 \sin\left(ax + b + \dfrac{\pi}{2} + \dfrac{\pi}{2}\right)$

$= a^2 \sin\left(ax + b + 2 . \dfrac{\pi}{2}\right)$

$y_3 = a^3 \cos\left(ax + b + 2 . \dfrac{\pi}{2}\right)$

$= a^3 \sin\left(ax + b + 2 . \dfrac{\pi}{2} + \dfrac{\pi}{2}\right) = a^3 \sin\left(ax + b + 3 . \dfrac{\pi}{2}\right)$

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$$\mathbf{y_n = a^n \sin\left(ax + b + n . \frac{\pi}{2}\right).}$$

$\left[\textbf{Cor.} \text{ If } b = 0, y = \sin ax, y_n = a^n \sin\left(ax + n\dfrac{\pi}{2}\right)\right]$

NOTES

**VII. The nth differential co-efficient of cos (ax + b)**

Let $y = \cos(ax + b)$

then $y_1 = -\sin(ax + b) \,.\, a = a \cos\left(ax + b + \dfrac{\pi}{2}\right)$ [Note this step]

$$\left[\because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right]$$

$$y_2 = -a^2 \sin\left(ax + b + \frac{\pi}{2}\right) = a^2 \cos\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^2 \cos\left(ax + b + 2\,.\,\frac{\pi}{2}\right)$$

$$y_3 = -a^3 \sin\left(ax + b + 2\,.\,\frac{\pi}{2}\right) = a^3 \cos\left(ax + b + 2\,.\,\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= a^3 \cos\left(ax + b + 3\,.\,\frac{\pi}{2}\right)$$

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$$\mathbf{y_n = a^n \cos\left(ax + b + n\,\frac{\pi}{2}\right).}$$

[**Cor.** If $y = \cos ax$

$$y_n = a^n \cos\left(ax + n\,.\,\frac{\pi}{2}\right)\Bigg]$$

**VIII. The nth differential co-efficient of $e^{ax}$ sin (bx + c)**

Let $y = e^{ax} \sin(bx + c)$

then $y_1 = e^{ax}\dfrac{d}{dx}[\sin(bx + c)] + \sin(bx + c)\dfrac{d}{dx}[e^{ax}]$

$$= e^{ax} \,.\, \cos(bx + c) \,.\, b + \sin(bx + c) \,.\, a \,.\, e^{ax}$$

$$= e^{ax}[a \sin(bx + c) + b \cos(bx + c)].$$

We determine two constants $r$ and $\theta$, to change the expression into a single sine which will enable us to make the required generalisation by putting,

$$\mathbf{a = r \cos\theta, \; b = r \sin\theta}$$

$\therefore$ $r = \sqrt{a^2 + b^2}, \; \theta = \tan^{-1}\dfrac{b}{a}.$

Hence $y_1 = e^{ax}[r\cos\theta \,.\, \sin(bx + c) + r \sin\theta \,.\, \cos(bx + c)]$

$$= r\, e^{ax} \sin(bx + c + \theta).$$

Thus $y_1$ is obtained from $y$ on multiplying it by the constant $r$ and increasing the angle by the constant $\theta$. Repeating the same rule to $y_1$, we have

$$y_2 = r^2 e^{ax} \sin(bx + c + 2\theta).$$

Similarly, $y_3 = r^3 e^{ax} \sin(bx + c + 3\theta).$

Hence, in general, $y_n = \dfrac{d^n}{dx^n}[e^{ax} \sin(bx + c)] = r^n \,.\, e^{ax} \sin(bx + c + n\theta)$

Putting values of $r$ and $\theta$,

$$\mathbf{y_n = (a^2 + b^2)^{n/2} e^{ax} \,.\, \sin\left(bx + c + n\,.\,\tan^{-1}\frac{b}{a}\right).}$$

NOTES

**IX. The nth differential co-efficient of $e^{ax}$ cos (bx + c)**

Let $y = e^{ax} \cos (bx + c)$

then $y_1 = a\, e^{ax} \cos (bx + c) - b\, e^{ax} \sin (bx + c)$

$= e^{ax} [\, a \cos (bx + c) - b \sin (bx + c)]$

As in Art. VIII, put $a = r \cos \theta$, and $b = r \sin \theta$.

$\therefore$ $r = \sqrt{a^2 + b^2}$, $\tan \theta = \dfrac{b}{a}$, *i.e.,* $\theta = \tan^{-1} \dfrac{b}{a}$

$\therefore$ $y_1 = e^{ax}\, r\, [\cos \theta \,.\, \cos (bx + c) - \sin \theta \sin (bx + c)]$

$= r\, e^{ax} [\cos (bx + c + \theta)]$.

Thus $y_1$ is obtained from $y$ on multiplying it by the constant $r$ and increasing the angle by the constant $\theta$.

Repeating the same rule to $y_1$, we have

$y_2 = r^2 e^{ax} \cos (bx + c + 2\theta)$.

Similarly, $y_3 = r^3 e^{ax} \cos (bx + c + 3\theta)$.

Hence, in general, $y_n = \dfrac{d^n}{dx^n} [\, e^{ax} \cos (bx + c)] = r^n e^{ax} \,.\, \cos (bx + c + n\theta)$

Putting values of $r$ and $\theta$,

$$\mathbf{y_n = (a^2 + b^2)^{n/2}\, e^{ax} \,.\, \cos \left( bx + c + n \,.\, \tan^{-1} \frac{b}{a} \right).}$$

## SOLVED EXAMPLES

**Example 6.** *Find the nth differential co-efficient of log $(ax + x^2)$.*

**Solution.** Let $y = \log (ax + x^2) = \log x(a + x)$

$= \log x + \log (x + a)$

Differentiating $n$ times,

$$y_n = \frac{d^n}{dx^n} \log x + \frac{d^n}{dx^n} \log (x + a)$$

$$= \frac{(-1)^{n-1} (n-1)\,!\,.\,1^n}{x^n} + \frac{(-1)^{n-1} (n-1)\,!\,.\,1^n}{(x + a)^n} \quad \text{[By Art. IV]}$$

$$= (-1)^{n-1} (n-1)\,! \left[ \frac{1}{x^n} + \frac{1}{(x + a)^n} \right].$$

**Example 7.** *If $y = \cos^3 x$, find $y_n$.*

**Solution.** We know that

$$\cos 3x = 4 \cos^3 x - 3 \cos x$$

$$\cos^3 x = \frac{1}{4} [\cos 3x + 3 \cos x]$$

*i.e.,* $$y = \frac{1}{4} [\cos 3x + 3 \cos x]$$

Differentiating $n$ times,

$$y_n = \frac{1}{4} \left[ \frac{d^n}{dx^n} \cos 3x + 3 \frac{d^n}{dx^n} \cos x \right]$$

$$= \frac{1}{4}\left[3^n \cos\left(3x + \frac{n\pi}{2}\right) + 3.1^n \cos\left(x + \frac{n\pi}{2}\right)\right] \quad \text{[By Art. VII]}$$

$$= \frac{1}{4}\left[3^n \cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right].$$

NOTES

**Example 8.** *If $y = \sin x \sin 3x$, find $y_n$.*

**Solution.** $y = \sin x \sin 3x$

$$= \frac{1}{2}[2 \sin 3x \sin x] = \frac{1}{2}[\cos 2x - \cos 4x]$$

$[\because \; 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{d^n}{dx^n}\cos 2x - \frac{d^n}{dx^n}\cos 4x\right]$$

$$= \frac{1}{2}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) - 4^n \cos\left(4x + \frac{n\pi}{2}\right)\right].$$

**Example 9.** *Find the nth differential co-efficient of $e^x \sin^3 x$.*

**Solution.** We know that,

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$\therefore \quad \sin^3 x = \tfrac{1}{4}[3 \sin x - \sin 3x] \qquad ...(1)$$

Let $y = e^x \sin^3 x = e^x \,.\, \tfrac{1}{4}[3 \sin x - \sin 3x]$ | From (1)

$$= \tfrac{3}{4}e^x \sin x - \tfrac{1}{4}\,.\, e^x \sin 3x$$

$$\therefore \quad y_n = \tfrac{3}{4}(1^2 + 1^2)^{n/2} \,.\, e^x \sin (x + n \tan^{-1} \tfrac{1}{1})$$

$$- \tfrac{1}{4}\,.\,(1^2 + 3^2)^{n/2}\, e^x \sin [3x + n \tan^{-1} \tfrac{3}{1}]$$

| Comparing to result in Art. (VIII)

$$= \frac{3}{4}\,.\,2^{n/2} e^x \sin\left(x + n\frac{\pi}{4}\right) - \frac{1}{4}\,.\,10^{n/2}\,.\,e^x \sin (3x + n \tan^{-1} 3).$$

**Example 10.** *If $y = \sqrt{x + a}$, find $y_n$.*

**Solution.** Here, $y = (x + a)^{1/2}$

$$\therefore \quad y_1 = \tfrac{1}{2}(x + a)^{-1/2}$$

$$y_2 = (\tfrac{1}{2})(-\tfrac{1}{2})(x + a)^{-3/2}$$

$$y_3 = \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})(x + a)^{-5/2} = (-1)^2 \frac{1.3}{2^3}(x + a)^{-5/.2}$$

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$$\therefore \quad y_n = (-1)^{n-1}\frac{1\,.\,3\,.\,5 \ldots \text{upto } (n-1) \text{ terms}}{2^n}(x + a)^{-\left(\frac{2n-1}{2}\right)}$$

$$= (-1)^{n-1}\frac{1.3 \ldots (2n-3)}{2^n}(x + a)^{-\left(\frac{2n-1}{2}\right)}.$$

**Remark.** If none of the formulae of the IX Articles is applicable to find $y_n$ in a problem, then we proceed as in the articles, *i.e.*, find $y_1$, $y_2$, $y_3$ and then generalise as done in the above example 6.

## EXERCISE B

NOTES

*Find the nth derivative of the following :*

**1.** (*i*) $4^x$ (*ii*) $5^x$ (*iii*) $e^{2x} + e^{-2x}$ (*iv*) $\log 3x$.

**2.** (*i*) $\dfrac{1}{(3-2x)^3}$ (*ii*) $\dfrac{1}{(bx+a)^2}$.

**3.** (*i*) $\log \sqrt{\dfrac{2x+1}{x-2}}$ (*ii*) $\log \sqrt{\dfrac{3-2x}{5+4x}}$ (*iii*) $\log (x^2 - a^2)$ (*iv*) $\dfrac{3x+7}{x+2}$.

**4.** (*i*) $\sin^2 x$ (*ii*) $\cos^2 x$ (*iii*) $\sin^3 x$ (*iv*) $\cos^4 x$.

[**Hint.** (*iv*)
$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}$$
$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}(1+\cos 4x) = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.]$$

(*v*) $\sin^4 x$.

**5.** (*i*) $\sin x \cos 2x$ (*ii*) $\cos x \cos 2x$.

**6.** (*i*) $\cos ax \sin bx$ (*ii*) $\sin x \sin 2x$ (*iii*) $\sin^3 x \cos^2 x$.

**7.** (*i*) $e^x \cos x$ (*ii*) $e^x \sin x$

(*iii*) $e^x \sin x \cos x$ (*iv*) $e^{3x} \sin^2 2x$.

**8.** (*i*) $e^{ax} \cos^2 x \sin x$ (*ii*) $e^x \cos^3 x$ (*iii*) $e^x \sin^2 x \sin 2x$

(*iv*) $a^x \cos x$. [**Hint.** $a^x = e^{\log a^x} = e^{x \log a}$]

**9.** If $\mu = \sin nx + \cos nx$, prove that $\mu_r = n^r [1 + (-1)^r \sin 2nx]^{1/2}$ where $\mu_r$ denotes the $r$th differential co-efficient of $\mu$ w.r.t. $x$.

[**Hint.**
$$\mu_r = n^r \sin\left(nx + \frac{r\pi}{2}\right) + n^r \cos\left(nx + \frac{r\pi}{2}\right)$$
$$= n^r\left[\sin\left(nx + \frac{r\pi}{2}\right) + \cos\left(nx + \frac{r\pi}{2}\right)\right]$$
$$= n^r\sqrt{\left[\sin\left(nx + \frac{r\pi}{2}\right) + \cos\left(nx + \frac{r\pi}{2}\right)\right]^2}$$
$$= n^r\sqrt{\sin^2\left(nx + \frac{r\pi}{2}\right) + \cos^2\left(nx + \frac{r\pi}{2}\right) + 2\sin\left(nx + \frac{r\pi}{2}\right)\cos\left(nx + \frac{r\pi}{2}\right)}$$
$$= n^r\sqrt{1 + \sin 2\left(nx + \frac{r\pi}{2}\right)} \qquad \Big|\because\ 2\sin\theta\cos\theta = \sin 2\theta$$
$$= n^r\sqrt{1+\sin(r\pi + 2nx)} = n^r\sqrt{1+(-1)^r \sin 2nx}\Big].$$

### Answers

**1.** (*i*) $4^x [\log 4]^n$ (*ii*) $[\log 5]^n . 5^x$

(*iii*) $2^n[e^{2x} + (-1)^n e^{-2x}]$ (*iv*) $\dfrac{[-1]^{n-1}(n-1)!}{x^n}$.

**2.** (*i*) $\dfrac{(n+2)!\,.\,2^{n-1}}{[3-2x]^{n+3}}$ (*ii*) $\dfrac{[-1]^n(n+1)!\,.\,b^n}{[bx+a]^{n+2}}$.

**3.** (*i*) $\dfrac{[-1]^{n-1}(n-1)!}{2}\left[\dfrac{2^n}{[2x+1]^n}-\dfrac{1}{(x-2)^n}\right]$

(*ii*) $\dfrac{1}{2}(-1)^{n-1}(n-1)!\left[\dfrac{[-2]^n}{[3-2x]^n}-\dfrac{4^n}{[5+4x]^n}\right]$

(*iii*) $(-1)^{n-1}(n-1)!\left[\dfrac{1}{(x+a)^n}+\dfrac{1}{(x-a)^n}\right]$ (*iv*) $\dfrac{(-1)^n n!}{(x+2)^{n+1}}$.

**4.** (*i*) $-2^{n-1}\cos[2x+n\,.\,\pi/2]$ (*ii*) $2^{n-1}\cos[2x+n\,.\,\pi/2]$

(*iii*) $\dfrac{3}{4}\sin[x+n\,.\,\pi/2]-\dfrac{1}{4}.\,3^n\sin[3x+n\,.\,\pi/2]$

(*iv*) $2^{n-1}\cos[2x+n\,.\,\pi/2]+2^{2n-3}\cos[4x+n\,.\,\pi/2]$

(*v*) $-2^{n-1}\cos[2x+n\,.\,\pi/2]+2^{2n-3}\cos[4x+n\,.\,\pi/2]$.

**5.** (*i*) $\dfrac{1}{2}[3^n\sin(3x+n\,.\,\pi/2)-\sin(x+n\,.\,\pi/2)]$ (*ii*) $\dfrac{1}{2}\left[3^n\cos\left(3x+n\dfrac{\pi}{2}\right)+\cos\left(x+n\dfrac{\pi}{2}\right)\right]$.

**6.** (*i*) $\dfrac{1}{2}\left[(a+b)^n\sin\left\{(a+b)x+\dfrac{n\pi}{2}\right\}+(b-a)^n\sin\left\{(b-a)x+\dfrac{n\pi}{2}\right\}\right]$

(*ii*) $\dfrac{1}{2}[\cos(x+n\,.\,\pi/2)-3^n\cos(3x+n\,.\,\pi/2)]$

(*iii*) $\dfrac{1}{16}\left[2\sin\left(x+\dfrac{n\pi}{2}\right)-5^n\sin\left(5x+\dfrac{n\pi}{2}\right)+3^n\sin\left(3x+\dfrac{n\pi}{2}\right)\right]$.

**7.** (*i*) $2^{n/2}\,.\,e^x\cos[x+n\,.\,\pi/4]$ (*ii*) $2^{n/2}\,.\,e^x\sin[x+n\,.\,\pi/4]$

(*iii*) $\dfrac{1}{2}\,.\,5^{n/2}e^x\sin(2x+n\tan^{-1}2)$ (*iv*) $\dfrac{1}{2}\left[3^n e^{3x}-5^n e^{3x}\cos\left(4x+n\tan^{-1}\dfrac{4}{3}\right)\right]$.

**8.** (*i*) $\dfrac{1}{4}[(a^2+1)^{n/2}e^{ax}\sin(x+n\tan^{-1}1/a)+(a^2+9)^{n/2}e^{ax}\sin(3x+n\tan^{-1}3/a)]$

(*ii*) $\dfrac{3}{4}\,.\,2^{n/2}\,.\,e^x\cos[x+n\,.\,\pi/4]+\dfrac{1}{4}\,.\,10^{n/2}e^x\cos[3x+n\tan^{-1}3]$

(*iii*) $\dfrac{1}{2}\,.\,5^{n/2}\,.\,e^x\sin[2x+n\tan^{-1}2]-\dfrac{1}{4}[17]^{n/2}\,.\,e^x\sin[4x+n\tan^{-1}4]$

(*iv*) $a^x[(1+\log a)^2]^{n/2}\,.\,\cos[x+n\tan^{-1}1/\log a]$.

## USE OF PARTIAL FRACTIONS

In order to determine the *n*th derivative, of any rational function, we have to split it into partial fractions.

Forming partial fractions is converse process of taking L.C.M.

**To resolve a fraction into partial fractions, the degree of numerator must be less than the degree of denominator.**

Partial fractions for :

(*i*) $\dfrac{f(x)}{(x-a)(x-b)(x-c)}$ are $\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$

(*ii*) $\dfrac{f(x)}{(x-a)^2(x-b)}$ are $\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$

NOTES

(*iii*) $\dfrac{f(x)}{(x-a)^3(x-b)}$ are $\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-a)^3}+\dfrac{D}{x-b}$

(*iv*) $\dfrac{f(x)}{(x-a)(x-b)(px^2+qx+r)}=\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{Cx+D}{px^2+qx+r}$.

To find A, B, C, D, etc., we put each linear factor of L.C.M. equal to zero. The remaining constants are obtained by comparing coefficients of like powers of $x$ on both sides.

## SOLVED EXAMPLES

**Example 11.** *Find nth derivative of* $\dfrac{1}{x^2-a^2}$.

**Solution.** Let $y=\dfrac{1}{x^2-a^2}=\dfrac{1}{(x-a)(x+a)}=\dfrac{A}{x-a}+\dfrac{B}{x+a}$ ...(1)

Multiplying both sides by L.C.M. = $(x-a)(x+a)$,

$$1 = A(x+a)+B(x-a).$$

Put $x-a=0$, *i.e.*, $x=a$,

$\therefore$ $1 = A(a+a)+B(a-a)$ or $1=2aA$ $\therefore$ $A=\dfrac{1}{2a}$.

Put $x+a=0$, *i.e.*, $x=-a$.

$\therefore$ $1=A(-a+a)+B(-a-a)$, or $1=-2aB$ $\therefore$ $B=\dfrac{-1}{2a}$.

Putting values of A and B in (1),

$$y=\frac{\frac{1}{2a}}{x-a}-\frac{\frac{1}{2a}}{x+a}=\frac{1}{2a}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$\therefore \quad y_n=\frac{1}{2a}\left[\frac{d^n}{dx^n}\frac{1}{x-a}-\frac{d^n}{dx^n}\frac{1}{x+a}\right]$$

$$=\frac{1}{2a}\left[\frac{(-1)^n n!\,1^n}{(x-a)^{n+1}}-\frac{(-1)^n n!\,.\,1^n}{(x+a)^{n+1}}\right]$$

$$\left(\because \text{ By Article III, } \frac{d^n}{dx^n}\frac{1}{ax+b}=\frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}\right)$$

$$=\frac{1}{2a}(-1)^n n!\left[\frac{1}{(x-a)^{n+1}}-\frac{1}{(x+a)^{n+1}}\right].$$

**Example 3.** *Prove that the value of the nth differential co-efficient of* $\dfrac{x^3}{x^2-1}$ *for x = 0 is zero, if n is even ; and is* – (*n* !) *if n is odd and greater than 1.*

**Solution.** Let $y=\dfrac{x^3}{x^2-1}=\dfrac{x^3}{(x-1)(x+1)}$

$$x^2-1)\ x^3\ (x$$
$$\underline{\ x^3 - x\ }$$
$$x$$

or $y=\dfrac{x^3}{(x-1)(x+1)}=x+\dfrac{A}{x-1}+\dfrac{B}{x+1}$ ...(1)

Multiplying by L.C.M. = $(x - 1)(x + 1)$

$$x^3 = x(x-1)(x+1) + A(x+1) + B(x-1)$$

Put $x = 1$, $\quad 1 = A(2) \quad$ or $\quad A = \frac{1}{2}$

Put $x = -1$, $\quad -1 = B(-2) \quad$ or $\quad B = \frac{1}{2}$.



Putting values of A and B in (1), we get

$$y = x + \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right].$$

If $n > 1$, the second and higher derivatives of $x$ are 0.

$\therefore$ For $n > 1$,

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+1)^{n+1}} + \frac{(-1)^n n!}{(x-1)^{n+1}}\right]$$

$$= \frac{1}{2}(-1)^n n!\left[\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}}\right] \qquad ...(1)$$

**Case I.** When $n$ is even, putting $x = 0$ in (1),

$$(y_n)_0 = \frac{(+1)n!}{2}\left[\frac{1}{1} + \frac{1}{(-1)}\right] = \frac{n!}{2}[1 - 1] = 0.$$

**Case II.** When $n$ is odd,

$$(y_n)_0 = \frac{(-1)n!}{2}\left[\frac{1}{1} + \frac{1}{(+1)}\right] = -\frac{(n!)}{2}[1 + 1] = -n!.$$

## EXERCISE C

*Find the nth derivative of the following :*

**1.** (*i*) $\frac{1}{(x+2)(x-1)}$ (*ii*) $\frac{x}{1 + 3x + 2x^2}$.

**2.** (*i*) $\frac{x^2}{(x+2)(2x+3)}$ (*ii*) $\frac{5x + 12}{x^2 + 5x + 6}$ (*iii*) $\frac{x+1}{6x^2 - 7x - 3}$.

**3.** If $y = \frac{x+1}{x^2 - 4}$, $x \neq \pm 2$, prove that $y_n = \frac{(-1)^n n!}{4}\left[\frac{3}{(x-2)^{n+1}} + \frac{1}{(x+2)^{n+1}}\right]$.

**4.** Find the $n$th derivative of :

(*i*) $\frac{1}{(x^2 - a^2)(x^2 - b^2)}$ (*ii*) $\frac{x^2}{(x-1)^3(x+1)}$.

### Answers

**1.** (*i*) $\frac{[-1]^n \cdot n!}{3}\left[\frac{1}{(x-1)^{n+1}} - \frac{1}{(x+2)^{n+1}}\right]$ (*ii*) $(-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{2^n}{(2x+1)^{n+1}}\right]$

**2.** (*i*) $\frac{[-1]^n n!}{2}\left[\frac{9 \cdot 2^n}{(2x+3)^{n+1}} - \frac{8}{(x+2)^{n+1}}\right]$ (*ii*) $[-1]^n n!\left[\frac{2}{(x+2)^{n+1}} + \frac{3}{(x+3)^{n+1}}\right]$

(*iii*) $\frac{[-1]^n n!}{11}\left[\frac{2^n \cdot 5}{(2x-3)^{n+1}} - \frac{3^n \cdot 2}{(3x+1)^{n+1}}\right]$.

4. (i) $\dfrac{[-1]^n n!}{a^2-b^2}\left[\dfrac{1}{2a}\left\{\dfrac{1}{(x-a)^{n+1}}-\dfrac{1}{(x+a)^{n+1}}\right\}-\dfrac{1}{2b}\left\{\dfrac{1}{(x-b)^{n+1}}-\dfrac{1}{(x+b)^{n+1}}\right\}\right]$

(ii) $[-1]^n\, n!\left[\dfrac{(n+2)(n+1)}{4(x-1)^{n+3}}+\dfrac{3(n+1)}{4(x-1)^{n+2}}+\dfrac{1}{8(x-1)^{n+1}}-\dfrac{1}{8(x+1)^{n+1}}\right].$

NOTES

## LEIBNITZ'S THEOREM

This theorem helps us to find the $n$th differential co-efficient of the product of two functions in terms of the successive derivatives of the functions.

**Statement. If u, v be two functions of x, having derivatives of the nth order, then**

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n uv_n,$$

**where suffixes of u and v denote differentiations w.r.t. x.**

**Proof.** We shall prove the theorem by Mathematical Induction.

**Step I.** Let $y = uv$

By actual differentiation, we have

$$y_1 = u_1 v + uv_1$$

and $$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + uv_2$$

$$= u_2 v + 2u_1 v_1 + uv_2 = u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 uv_2.$$

Thus, the theorem is true for $n = 1, 2$.

**Step II.** Let us assume that the theorem is true for a particular value of $n$, say $m$, so that, we have

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots\ldots\ldots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \ldots\ldots\ldots + {}^mC_m uv_m.$$

**Step III.** Differentiating both sides, we have

$$\begin{aligned} y_{m+1} &= u_{m+1} v + u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1} v_2 \\ &\quad + {}^mC_2 u_{m-1} v_2 + {}^mC_2 u_{m-2} v_3 + \ldots\ldots. \\ &\quad + {}^mC_{r-1} u_{m-r+2} v_{r-1} + {}^mC_{r-1} u_{m-r+1} v_r \\ &\quad + {}^mC_r u_{m-r+1} v_r + {}^mC_r u_{m-r} v_{r+1} + \ldots\ldots\ldots \\ &\quad + {}^mC_m u_1 v_m + {}^mC_m uv_{m+1} \\ &= u_{m+1} v + ({}^mC_1 + 1) v_m v_1 + ({}^mC_2 + {}^mC_1) u_{m-1} v_2 + \ldots\ldots \\ &\quad + ({}^mC_r + {}^mC_{r-1}) u_{m-r+1} v_r + \ldots\ldots + {}^mC_m u\, v_{m+1}. \end{aligned}$$

But, we know that

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$$

Putting $r = 1, 2, 3, \ldots\ldots\ldots.$

$${}^mC_0 + {}^mC_1 = {}^{m+1}C_1 \quad \text{or} \quad 1 + {}^mC_1 = {}^{m+1}C_1$$

$${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$$

..................................................

and $${}^mC_m = 1 = {}^{m+1}C_{m+1}.$$

$\therefore \quad y_{m+1} = u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + \ldots\ldots$
$+ {}^{m+1}C_r u_{m-r+1} v_r + \ldots\ldots.. + {}^{m+1}C_{m+1} uv_{m+1}$

which is of exactly the same form as the theorem to be proved with $n = m + 1$.

$\therefore$ If the theorem is true for $n = m$, then it is also true for the next higher value $n = m + 1$.

But in Step I, we have proved that the theorem is true for $n = 1$ and $n = 2$.

$\therefore$ it must be true for next higher value $n = 2 + 1 = 3$.

Again $\because$ the theorem is true for $n = 3$.

$\therefore$ it must be true for next higher value $n = 3 + 1$, *i.e.*, 4 and so on.

Hence the theorem is true for any positive integer $n$.

## SOLVED EXAMPLES

**Example 13.** *Find the nth derivative of $x^2 \sin x$.*

**Solution.** Let $u = \sin x$ and $v = x^2$

| | |
|---|---|
| $\therefore \quad u_n = \sin\left(x + n\dfrac{\pi}{2}\right)$ | $v_1 = 2x$ |
| $u_{n-1} = \sin\left[x + (n-1)\dfrac{\pi}{2}\right]$ | $v_2 = 2$ |
| $u_{n-2} = \sin\left[x + (n-2)\dfrac{\pi}{2}\right]$ | $v_3 = 0$ |

Now by Leibnitz's Theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2$$

or $$\frac{d^n}{dx^n}(x^2 \sin x) = \sin\left(x + n\frac{\pi}{2}\right).\, x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right] 2x + {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right] 2$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1) \sin\left[x + (n-2)\frac{\pi}{2}\right].$$

**Note.** Generally, we take $x^n$ as $v$.

**Example 4.** *If $y = \sin(m \sin^{-1} x)$, then prove that*

$$(1 - x^2)y_2 - xy_1 + m^2 y = 0$$

*and* $$(1 - x^2)y_{n+2} = (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n.$$

**Sol.** $$y = \sin(m \sin^{-1} x) \qquad \ldots(1)$$

$$\therefore \quad y_1 = \cos(m \sin^{-1} x)\,.\, \frac{m}{\sqrt{1 - x^2}}$$

Cross-multiplying $\sqrt{1 - x^2}\, y_1 = m \cos(m \sin^{-1} x)$.

Squaring both sides,

$$(1 - x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x)$$
$$= m^2[1 - \sin^2(m \sin^{-1} x)] \qquad | \because \cos^2\theta = 1 - \sin^2\theta$$
$$= m^2(1 - y^2) \qquad \text{[By (1)]}$$

Again differentiating both sides w.r.t. $x$,

$$(1 - x^2)\frac{d}{dx}y_1^2 + y_1^2 \frac{d}{dx}(1 - x^2) = m^2\left[0 - \frac{d}{dx}y^2\right]$$

or $$(1 - x^2)\, 2y_1 y_2 - 2xy_1^2 = -2m^2 yy_1$$

NOTES

Dividing every term by $2y_1$,

$$(1-x^2)y_2 - xy_1 = -m^2y$$

or $$(1-x^2)y_2 - xy_1 + m^2y = 0.$$

Now differentiating every term $n$ times by Leibnitz's Theorem, we have

$$(y_2)_n (1-x^2) + {}^nC_1 (y_2)_{n-1}(-2x) + {}^nC_2 (y_2)_{n-2}(-2)$$
$$-[(y_1)_n x + {}^nC_1 (y_1)_{n-1} . 1] + m^2y_n = 0$$

or $$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}(-2x) + {}^nC_2 y_n(-2) - y_{n+1} . x - {}^nC_1 y_n + m^2y_n = 0$$

or $$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

or $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Hence $$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n.$$

**Example 15.** *If $y^{1/m} + y^{-1/m} = 2x$, prove that*

$$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0.$$

**Solution.** $$y^{1/m} + y^{-1/m} = 2x$$

Put $$y^{1/m} = z$$

$$\therefore \quad z + \frac{1}{z} = 2x \quad \text{or} \quad z^2 - 2xz + 1 = 0 \qquad \therefore \quad z = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\therefore \quad z = \frac{2x \pm 2\sqrt{x^2-1}}{2} = x \pm \sqrt{x^2-1}$$

*i.e.*, $$y^{1/m} = x \pm \sqrt{x^2-1} \quad \text{or} \quad y = (x \pm \sqrt{x^2-1})^m$$

If $$y = (x + \sqrt{x^2-1})^m, \text{ then}$$

$$y_1 = m(x+\sqrt{x^2-1})^{m-1}\left(1 + \frac{1}{2} . \frac{1}{\sqrt{x^2-1}} . 2x\right)$$

$$= m(x+\sqrt{x^2-1})^{m-1}\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$= \frac{m(x+\sqrt{x^2-1})^m}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

And, similarly, if $y = (x - \sqrt{x^2-1})^m$,

Then $$y_1 = \frac{-my}{\sqrt{x^2-1}}.$$

In either case, squaring, we get

$$y_1^2 = \frac{m^2y^2}{x^2-1} \quad \text{or} \quad (x^2-1)y_1^2 = m^2y^2.$$

Differentiating again,

$$(x^2-1) . 2y_1y_2 + y_1^2 . 2x = m^2 . 2yy_1$$

Dividing both sides by $2y_1$, we have

$$(x^2-1)y_2 + xy_1 - m^2y = 0$$

Differentiating $n$ times by Leibnitz's Theorem,

$$(y_2)_n(x^2-1)+{}^nC_1\,(y_2)_{n-1}\,(2x)+{}^nC_2\,(y_2)_{n-2}.\,2+(y_1)_n x+{}^nC_1\,(y_1)_{n-1}\,.\,1-m^2y_n=0$$

or $$y_{n+2}\,(x^2-1)+n\,.\,y_{n+1}\,.\,2x+\frac{n(n-1)}{2!}\,.\,y_n\,.\,2+[y_{n+1}\,.\,x+ny_n\,.\,1]-m^2y_n=0$$

or $$(x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-n+n-m^2)y_n=0$$

or $$(x^2-1)y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)y_n=0.$$

## EXERCISE D

*Apply Leibnitz's Theorem to find $y_n$ in the following cases :*

**1.** (*i*) $x^3 e^{ax}$ (*ii*) $x^2 e^x$ (*iii*) $x^3 e^{2x}$

**2.** (*i*) $x^3 \cos x$ (*ii*) $x^3 \sin ax$

**3.** (*i*) $x^3 \log x$ (*ii*) $x^2 e^x \cos x$.

**4.** State Leibnitz's Theorem and hence or otherwise show that if $y = x^2 e^x$, then

$$\frac{d^8y}{dx^8}=28\frac{d^2y}{dx^2}-48\frac{dy}{dx}+21y.$$

**5.** If $y = x^n \log x$, prove that $y_{n+1}=\dfrac{n!}{x}$.

[**Hint.** We have $y_1 = x^n\,.\,\dfrac{1}{x}+nx^{n-1}\log x$. Multiplying by $x$, we get

$y_1x = x^n + nx^n \log x$ or $y_1x = x^n + ny$.

Now, diff. both sides $n$ times and use $\dfrac{d^n}{dx^n}(x^n)=n!$]

**6.** If $y = x^2 \sin x$, prove that $\dfrac{d^ny}{dx^n}=(x^2-n^2+n)\sin\left(x+\dfrac{n\pi}{2}\right)-2nx\cos\left(x+\dfrac{n\pi}{2}\right)$.

**7.** Differentiate $n$ times the equation :

(*i*) $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+a^2y=0$ (*ii*) $x^2y_2+xy_1+y=0$.

**8.** If $y = a\cos(\log x) + b\sin(\log x)$, show that :

$$x^2y_{n+2}+(2n+1)\,xy_{n+1}+(n^2+1)\,y_n=0.$$

**9.** If $y=\sin^{-1}x$, prove that $(1-x^2)y_{n+2}-(2n+1)\,xy_{n+1}-n^2y_n=0$.

**10.** If $y=[\log(x+\sqrt{1+x^2})]^2$, prove that

$$(1+x^2)y_{n+2}+(2n+1)\,xy_{n+1}+n^2y_n=0.$$

**11.** (*a*) If $y=e^{m\sin^{-1}x}$, prove that :

(*i*) $(1-x^2)y_2-xy_1=m^2y$ (*ii*) $(1-x^2)y_{n+2}-(2n+1)\,xy_{n+1}-(n^2+m^2)y_n = 0$.

(*b*) If $x=\sin\left(\dfrac{\log y}{a}\right)$, prove that $(1-x^2)\,y_{n+2}-(2n+1)x\,y_{n+1}-(n^2+a^2)\,y_n=0$.

(*c*) If $y=e^{m\cos^{-1}x}$, show that $(1-x)^2\,y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$.

**12.** If $\cos^{-1}\left(\dfrac{y}{b}\right)=\log\left(\dfrac{x}{n}\right)^n$, prove that $x^2y_{n+2}+(2n+1)x\,y_{n+1}+2n^2y_n=0$.

**13.** If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that $(1-x^2)y_{n+2}-(2n+3)\,x\,y_{n+1}-(n+1)^2\,y_n=0$.

**14.** If $y = (x^2 - 1)^n$, prove that $(x^2 - 1)\, y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$.

**15.** If $y = e^{\tan^{-1} x}$ (or $x \tan (\log y)$), prove that

$$(1 + x^2)\, y_{n+2} + [(2n+2)x - 1]y_{n+1} + n(n+1)\, y_n = 0.$$



## Answers

**1.** (*i*) $e^{ax} \,.\, a^{n-3} [a^3x^3 + 3n \,.\, a^2x^2 + 3n(n-1)ax + n(n-1)(n-2)]$

(*ii*) $e^x [x^2 + 2nx + n(n-1)]$ (*iii*) $2^{n-3} e^{2x} [8x^3 + 12nx^2 + 6n(n-1)x + n(n-1)(n-2)]$.

**2.** (*i*) $x^3 \cos\left(x + \frac{n}{2}\pi\right) + 3nx^2 \cos\left(x + \frac{n-1}{2}\pi\right) + 3n(n-1)\, x \cos\left(x + \frac{n-2}{2}\pi\right)$

$$+ n(n-1)(n-2) \cos\left(x + \frac{n-3}{2}\pi\right).$$

(*ii*) $a^{n-3}\left[a^3 x^3 \sin\left(ax + \frac{n\pi}{2}\right) + 3na^2x^2 \sin\left\{ax + (n-1)\frac{\pi}{2}\right\}\right.$

$$\left. + 3n\,(n-1)\, ax \sin\left\{ax + (n-2)\,.\,\frac{\pi}{2}\right\} + n(n-1)(n-2) \sin\left\{ax + (n-3)\frac{\pi}{2}\right\}\right].$$

**3.** (*i*) $\frac{[-1]^{n-1} n\,!}{x^{n-3}}\left[\frac{1}{n} - \frac{3}{n-1} + \frac{3}{n-2} - \frac{1}{n-3}\right]$.

(*ii*) $e^x\left[2^{n/2} x^2 \cos\left(x + \frac{n\pi}{4}\right) + 2^{\frac{n+1}{2}} nx \cos\left(x + \overline{n-1}\,\frac{\pi}{4}\right) + 2^{\frac{n-2}{2}} n(n-1) \cos\left(x + \overline{n-2}\,\frac{\pi}{4}\right)\right]$.

**7.** (*i*) $[1 - x^2]y_{n+2} - [2n+1]xy_{n+1} - [n^2 - a^2]y_n = 0$

(*ii*) $x^2y_{n+2} + [2n+1]\, xy_{n+1} + [n^2+1]y_n = 0$.

## Hints and Solutions

**3.** (*ii*) Let $u = e^x \cos x$ and $v = x^2$.

$$u_n = (a^2 + b^2)^{n/2} e^x \cos\left(x + n \tan^{-1}\frac{b}{a}\right)$$

Here $a = 1$ and $b = 1$

$\therefore$ $a^2 + b^2 = 1 + 1 = 2$ and $\tan^{-1}\frac{b}{a} = \tan^{-1} 1 = \frac{\pi}{4}$

$\therefore$ $u_n = 2^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right)$.

**5.** $y = x^n \log x$ ...(1)

Diff. w.r.t. $x$, $y_1 = x^n \,.\, \frac{1}{x} + n\, x^{n-1} \log x$

Multiplying every term by $x$,

$y_1\, x = x^n + n\, x^n \log x$ or $y_1\, x = x^n + n\, y$ [by (1)]

Diff. both sides $n$ times w.r.t. $x$,

$${}^nC_0\, (y_1)_n \,.\, x + {}^nC_1\, (y_1)_{n-1} \,.\, 1 = n\,! + n\, y_n \qquad \left(\because \frac{d^n}{dx^n}(x^n) = n\,!\right)$$

or $y_{n+1}\, x + n\, y_n = n\,! + n\, y_n$ or $y_{n+1}\, x = n\,!$

$\therefore$ $y_{n+1} = \frac{n\,!}{x}$.

**NOTES**

**6.** $$y_n = x^2 \frac{d^n}{dx^n}(\sin x) + {}^nC_1 \,.\, 2x \frac{d^{n-1}}{dx^{n-1}}(\sin x) + {}^nC_2 \,.\, 2. \frac{d^{n-2}}{dx^{n-2}}(\sin x) \quad ...(1)$$

Now $\frac{d^n}{dx^n}(\sin x) = 1^n \sin\left(\frac{n\pi}{2} + x\right) = \sin\left(\frac{n\pi}{2} + x\right)$ ...(2)

Changing $n$ to $(n - 1)$, $\frac{d^{n-1}}{dx^{n-1}}(\sin x) = \sin\left((n-1)\frac{\pi}{2} + x\right)$

$$= \sin\left(\frac{n\pi}{2} - \frac{\pi}{2} + x\right) = \sin - \left(\frac{\pi}{2} - \frac{n\pi}{2} - x\right) = -\sin\left[\frac{\pi}{2} - \left(\frac{n\pi}{2} + x\right)\right] = -\cos\left(\frac{n\pi}{2} + x\right)$$

Changing $n$ to $(n - 2)$ in (2)

$$\frac{d^{n-2}}{dx^{n-2}}(\sin x) = \sin\left[(n-2)\frac{\pi}{2} + x\right]$$

$$= \sin\left(\frac{n\pi}{2} - \pi + x\right) = \sin\left[-\left(\pi - \frac{n\pi}{2} + x\right)\right] = -\sin\left(\frac{n\pi}{2} + x\right).$$

**11.** (*b*) $$x = \sin\left(\frac{\log y}{a}\right) \Rightarrow \sin^{-1}x = \frac{\log y}{a}$$

$$\Rightarrow \quad \log y = a \sin^{-1} x \quad \Rightarrow \quad y = e^{a \sin^{-1} x}.$$

**12.** $$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$$

$$\therefore \quad \frac{y}{b} = \cos \log\left(\frac{x}{n}\right)^n$$

$$\therefore \quad y = b \cos\left(n \log \frac{x}{n}\right).$$

**14.** $$y = (x^2 - 1)^n \quad ...(1)$$

Diff. both sides of (1) w.r.t. $x$,

$$y_1 = n(x^2 - 1)^{n-1} \,.\, 2x \text{ or } y_1 = 2nx \frac{(x^2 - 1)^n}{(x^2 - 1)}$$

Cross-multiplying $(x^2 - 1)\, y_1 = 2n\, x\, y$. [By (1)]

## DETERMINATION OF THE VALUE OF THE NTH DERIVATIVE OF A FUNCTION FOR x = 0

Sometimes it is required to find the $n$th derivative of a function for $x = 0$. The **working rule** to find $(y_n)_{x=0}$ is being given below :

**1.** Put the given function equal to $y$.

**2.** Find $y_1 = \frac{dy}{dx}$.

Then (*i*) Take L.C.M. (if possible).

(*ii*) Square both sides if square Roots are there.

(*iii*) Try to get $y$ in R.H.S. (if possible).

**3.** Again differentiate both sides w.r.t. $x$ to get an equation in $y_2$, $y_1$, $y$.

**4.** Differentiate both sides $n$ times w.r.t. $x$ by Leibnitz Theorem.

**Leibnitz Theorem** is

$$(uv)_n = u_n \,.\, v + {}^nC_1 \,.\, u_{n-1} \,.\, v_1 + {}^nC_2 \,.\, u_{n-2} \,.\, v_2 + ... + {}^nC_n\, uv_n.$$

5. Put $x = 0$ in equations of steps 1, 2, 3, 4.
6. Put $n = 1, 2, 3, 4$ in last equation of step 5.
7. Discuss the two cases when $n$ is even and when $n$ is odd.

NOTES

## SOLVED EXAMPLES

**Example 16.** *If $y = e^{m\cos^{-1}x}$, show that*

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 + m^2)\, y_n = 0$$

*and calculate $y_n(0)$.*

**Solution.** Here $\quad y = e^{m\cos^{-1}x} \qquad ...(1)$

$$\therefore \quad y_1 = e^{m\cos^{-1}x} \cdot \frac{-m}{\sqrt{1-x^2}} = -\frac{ym}{\sqrt{1-x^2}} \qquad ...(2)$$

Squaring both sides, we have

$$(1 - x^2)\, y_1^2 = m^2y^2$$

Differentiating w.r.t. $x$, we have

$$(1 - x^2)\, 2y_1y_2 - 2xy_1^2 = 2m^2yy_1$$

Dividing by $2y_1$, we get

$$(1 - x^2)\, y_2 - xy_1 = m^2y \qquad ...(3)$$

Differentiating $n$ times by Leibnitz's Theorem, we get

$$(1 - x^2)y_{n+2} - 2nxy_{n+1} - n(n - 1)y_n - xy_{n+1} - ny_n = m^2y_n$$

*i.e.,* $\quad (1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + m^2)y_n = 0 \qquad ...(4)$

By putting $x = 0$, in (1), (2), (3) and (4), we get

$$y(0) = e^{m\,.\,\pi/2}$$

$$y_1(0) = -m\,.\,e^{m\,.\,\pi/2}$$

$$y_2(0) = m^2\,.\,y(0) = m^2\,.\,e^{m\,.\,\pi/2}$$

$$y_{n+2}(0) = (n^2 + m^2)y_n(0) \qquad ...(5)$$

Putting $n = 1, 2, 3, 4$ ... in (5), we have

$$y_3(0) = (1^2 + m^2)y_1(0) = -m(1^2 + m^2)e^{m\,.\,\pi/2}$$

$$y_4(0) = (2^2 + m^2)y_2(0) = m^2(2^2 + m^2)e^{m\,.\,\pi/2}$$

$$y_5(0) = (3^2 + m^2)y_3(0) = -m(1^2 + m^2)(3^2 + m^2)e^{m\,.\,\pi/2}$$

$$y_6(0) = (4^2 + m^2)y_4(0) = m^2(2^2 + m^2)(4^2 + m^2)e^{m\,.\,\pi/2}$$

In general,

$$y_n(0) = \begin{cases} -m\,.\,e^{m\,.\,\pi/2}\,(1^2 + m^2)(3^2 + m^2)\ldots[(n-2)^2 + m^2] & \text{when } n \text{ is odd.} \\ m^2\,.\,e^{m\,.\,\pi/2}\,(2^2 + m^2)(4^2 + m^2)\ldots[(n-2)^2 + m^2], & \text{when } n \text{ is even.} \end{cases}$$

**Example 17.** *If $y = \tan^{-1} x$, prove that*

$$(1 + x^2)y_{n+1} + 2nxy_n + n(n - 1)\, y_{n-1} = 0.$$

*Hence determine the values of all the derivatives of $y$ with respect to $x$ when $x = 0$.*

**Sol.** Here $y = \tan^{-1} x$ ...(1)

$\therefore \quad y_1 = \dfrac{1}{1+x^2}$ ...(2)

or $y_1 (1 + x^2) = 1$

Differentiating $n$ times by Leibnitz's Theorem, we have

$$y_{n+1} (1 + x^2) + ny_n \,.\, 2x + \frac{n(n-1)}{2} y_{n-1} \,.\, 2 = 0$$

or $(1 + x^2) y_{n+1} + 2nxy_n + n(n-1) y_{n-1} = 0$ ...(3)

Putting $x = 0$ in (1), (2) and (3), we have

$$y(0) = 0, \; y_1 (0) = 1$$

and $y_{n+1} (0) = -n(n-1) y_{n-1} (0)$ ...(4)

Putting $n = 1, 2, 3, 4$ ......... in (4), we get

$$y_2 (0) = -1 \,.\, (0) \,.\, y(0) = 0$$
$$y_3 (0) = -2 \,.\, (1) \,.\, y_1 (0) = -2 = (-1)^1 \, 2\,!$$
$$y_4 (0) = -3 \,.\, (2) \,.\, y_2 (0) = 0$$
$$y_5 (0) = -4 \,.\, (3) \,.\, y_3 (0) = -4 \,.\, (3) \,.\, (-2) = (-1)^2 \, 4\,!$$

... ... ... ... ...

... ... ... ... ...

In general,

When $n$ is even, $y_n(0) = 0.$

When $n$ is odd, $y_n (0) = (-1)^{\frac{n-1}{2}} (n-1)\,!.$

## EXERCISE E

**1.** If $y = \sin^{-1} x$, prove that $(1 - x^2) y_{n+2} - (2n + 1)xy_{n+1} - n^2 y_n = 0.$
Also find the value of $y_n$ when $x = 0$.

**2.** If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $y_n = (n-1)^2 y_{n-2}$ for $x = 0$.

**3.** If $y = [x + \sqrt{1+x^2}\,]^m$, find $y_n (0)$.

**4.** Find $y_n (0)$, when $y = \log (x + \sqrt{1+x^2}\,)$.

**5.** (*i*) If $y = [\log (x + \sqrt{1+x^2}\,)]^2$, find all the derivatives of $y$ w.r.t. $x$, when $x = 0$.
(*ii*) If $y = (\sinh^{-1} x)^2$, prove that $(1 + x^2) y_{n+2} + (2n + 1) xy_{n+1} + n^2 y_n = 0.$
Hence find $y_n$ when $x = 0$.
[**Hint.** $\sinh^{-1} x = \log [x + \sqrt{1+x^2}\,]$].

**6.** If $y = e^{a \sin^{-1} x}$, prove that $(1 - x^2) y_{n+2} - (2n + 1) xy_{n+1} - (n^2 + a^2) y_n = 0.$
Deduce that $\displaystyle \mathop{\text{Lt}}_{x \to 0} \frac{y_{n+2}}{y_n} = n^2 + a^2$. Hence find $y_n(0)$.

**7.** If $y = \sin (m \sin^{-1} x)$, find $y_n(0)$.

8. If $y = x \log\left(\frac{x-1}{1+x}\right)$, prove that $y_m = (-1)^{m-2}(m-2)!\left[\frac{x-m}{(x-1)^m} - \frac{(x+m)}{(x+1)^m}\right]$.

NOTES

## Answers

1. When $n$ is even, $y_n(0) = 0$

   When $n$ is odd, $y_n(0) = 1^2 \,.\, 3^2 \,.\, 5^2 \,.....\, (n-2)^2$

3. When $n$ is even, $y_n(0) = m^2 (m^2 - 2^2)(m^2 - 4^2) \,.......\, [m^2 - (n-2)^2]$

   When $n$ is odd, $y_n(0) = m(m^2 - 1^2)(m^2 - 3^2) \,......\, [m^2 - (n-2)^2]$

4. When $n$ is even, $y_n(0) = 0$

   When $n$ is odd, $y_n(0) = (-1)^{\frac{n-1}{2}}\, 1^2 \,.\, 3^2 \,.\, 5^2 \,......\, (n-2)^2$

5. (*i*), (*ii*) When $n$ is even, $y_n(0) = (-1)^{\frac{n}{2}-1} \,.\, 2 \,.\, 2^2 \,.\, 4^2 \,.\, 6^2 \,...\, (n-2)^2$

   When $n$ is odd, $y_n(0) = 0$

6. When $n$ is even, $y_n(0) = a^2 (2^2 + a^2)(4^2 + a^2) \,..........\, [(n-2)^2 + a^2]$

   When $n$ is odd, $y_n(0) = a(1^2 + a^2)(3^2 + a^2) \,........\, [(n-2)^2 + a^2]$

7. When $n$ is even, $y_n(0) = 0$

   When $n$ is odd, $y_n(0) = m(1^2 - m^2)(3^2 - m^2) \,.....\, [(n-2)^2 - m^2]$.

# PARTIAL DIFFERENTIATION

NOTES

## STRUCTURE

Introduction
Differention of Partial Derivatives of First Order
Rules of Partial Differentiation
Def. Symmetric Function of $x$ and $y$
Partial Derivatives of Higher Order
Homogeneous Functions
If $u$ is a Homogeneous Function of $x$ and $y$ of Degree $n$, show that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are Homogeneous Functions of Degree $(n - 1)$ each
Euler's Theorem on Homogeneous Functions
If $u$ is a Homogeneous Functions in $x$ and $y$ of Degree $n$; then prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Composite Functions
Differentiation of Composite Functions
Change of Variables
Implicit Relation of $x$ and $y$
Differentiation of Implicit Equations
Theorem on Total Differentials

## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Differention of Partial Derivatives of First Order
- Rules of Partial Differentiation
- Def. Symmetric Function of $x$ and $y$
- Partial Derivatives of Higher Order
- Homogeneous Functions
- If $u$ is a Homogeneous Function of $x$ and $y$ of Degree $n$, show that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are Homogeneous Functions of Degree $(n - 1)$ each

## INTRODUCTION

NOTES

We know that **crops** are a function of rain, fertilizers, seeds etc.

Let us take five different fields in the same village.

Let us use different qualities of seeds in the five fields (All other factors being same).

Now we can know the effect of the quality of seeds on the yield or crops.

In calculus, we call it as the **Partial Derivative** of crops w.r.t. seeds.

**Note.** *Partial* means a 'part of '.

## DEFINITION OF PARTIAL DERIVATIVES OF FIRST ORDER

We know that the differential co-efficient of $f(x)$ w.r.t. $x$ is

$$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x) - f'(x)}{\delta x},$$

provided this limit exists, and is denoted by

$$f'(x) \quad \text{or} \quad \frac{d}{dx}[f(x)].$$

If $u = f(x, y)$ be a continuous function of two independent variables $x$ and $y$, then the differential co-efficient of $u$ w.r.t. $x$ (*regarding y as constant*) is called the *partial derivative or partial differential co-efficient of u* w.r.t. $x$ and is denoted by various symbols such as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y), f_x$$

Symbolically, if $u = f(x, y)$, then

$$\underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

if it exists, is called the *partial derivative or partial differential co-efficient of u* w.r.t. $x$ and is denoted by

$$\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x \quad \text{or} \quad u_x.$$

Similarly, by *keeping x constant* and allowing $y$ alone to vary, we can define the *partial derivative or partial differential co-efficient of u* w.r.t. $y$. It is denoted by any one of the symbols

$$\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), f_y$$

Symbolically, $$\frac{\partial u}{\partial y} = \underset{\delta y \to 0}{\text{Lt}} \frac{f(x, y+\delta y) - f(x, y)}{\delta y},$$

provided this limit exists.

Let $(a, b)$ be any point.

$\underset{h \to 0}{\text{Lt}} \frac{f(a+h, b) - f(a, b)}{h}$, if exists is called the **Partial Derivative** of the function $f$ w.r.t. $x$ at the point $(a, b)$ and is denoted by $\frac{\partial f}{\partial x}(a, b)$ or $f_x(a, b)$. Similarly, the **partial**

**derivative of f w.r.t. y** at the point $(a, b)$ is defined as $\underset{h \to 0}{\text{Lt}} \dfrac{f(a,b+h)-f(a,b)}{h}$, provided this limit exists

NOTES

and is denoted by $\dfrac{\partial f}{\partial y}$ $(a, b)$ or $f_y$ $(a, b)$.

For example, if $u = x^2 + 2xy + y^2$,

then $\dfrac{\partial u}{\partial x} = 2x + 2y$ and $\dfrac{\partial u}{\partial y} = 2x + 2y$.

Similarly, if $u = x^3 + 3x^2y + y^3$,

then $\dfrac{\partial u}{\partial x} = 3x^2 + 6xy$ and $\dfrac{\partial u}{\partial y} = 3x^2 + 3y^2$.

Similarly, if $u = f(x_1, x_2, x_3, ....., x_n)$, be a function of $n$ variables, then the partial differential co-efficient of $u$ with respect to $x_1$ is ordinary differential co-efficient of $u$ when all variables except $x_1$ are regarded as constants, and is denoted by any one of the symbols

$$\frac{\partial u}{\partial x_1}, \frac{\partial f}{\partial x_1}, fx_1.$$

**Two very useful first order partial derivatives :**

$$\frac{\partial}{\partial \mathbf{x}}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) = \mathbf{y}\frac{\partial}{\partial \mathbf{x}}\left(\frac{\mathbf{1}}{\mathbf{x}}\right) = \mathbf{y}\frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^{-1}) = \mathbf{y}(-1)\,\mathbf{x}^{-2} = -\frac{\mathbf{y}}{\mathbf{x}^2}$$

**and**

$$\frac{\partial}{\partial \mathbf{y}}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) = \frac{\mathbf{1}}{\mathbf{x}}\frac{\partial}{\partial \mathbf{y}}(\mathbf{y}) = \frac{\mathbf{1}}{\mathbf{x}}.$$

## RULES OF PARTIAL DIFFERENTIATION

**Rule 1.** If $u$ is a function of $x$, $y$ and we are to differentiate partially w.r.t. $x$, then, **y is treated as constant.**

Similarly, if we are to differentiate $u$ partially. w.r.t. $y$, then $x$ is treated as constant.

If $u$ is a function of $x, y, z$ and we are to differentiate partially w.r.t. $x$, then $y$ and $z$ are treated as constants.

**Rule 2.** If $z = u \pm v$, where $u$, $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \pm \frac{\partial v}{\partial y}.$$

**Rule 3.** If $z = uv$, where $u$, $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(uv) = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}.$$

**Rule 4.** If $z = \dfrac{u}{v}$, where $u$, $v$ are functions of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{u}{v}\right) = \frac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}.$$

**Rule 5.** If $z = f(u)$ where $u$ is a function of $x$ and $y$, then

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}.$$

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**Remark.** (*i*) If $z$ is a function of one variable $x$, we get $\frac{dz}{dx}$.

(*ii*) If $z$ is a function of two variables $x$ and $y$, we get $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(*iii*) If $z$ is a function of $n$ variables $x_1, x_2, ..., x_n$, we can find

$$\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \text{..........} \text{ and } \frac{\partial z}{\partial x_n}.$$

## DEF. SYMMETRIC FUNCTION OF x AND y

$u$ is said to be a symmetric function of $x$ and $y$ if on interchanging $x$ and $y$ ; (*i.e.*, changing $x$ to $y$ and $y$ to $x$) $u$ remains unchanged.

*Let us find the first order partial derivatives of the following :*

(*i*) $u = y^x$ *and* (*ii*) $u = \tan^{-1}\frac{x^2+y^2}{x-y}$.

(*i*) $u = y^x$ ...(*i*) (given)

Differentiating both sides of eqn. (*i*) partially w.r.t. $x$, (treating $y$ as constant)

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(y^x) = y^x \log y \qquad \because \frac{d}{dx}a^x = a^x \log a$$

Again differentiating both sides of eqn. (*i*) partially w.r.t. $y$, (treating $x$ as constant)

$$\frac{\partial u}{\partial y} = xy^{x-1} \qquad \because \frac{d}{dx}x^n = n\, x^{n-1}$$

(*ii*) $u = \tan^{-1}\frac{x^2+y^2}{x-y}$

Differentiating both sides partially w.r.t. $x$,

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x-y}\right)^2}\frac{\partial}{\partial x}\left(\frac{x^2+y^2}{x-y}\right) \qquad \because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x^2+y^2}{x-y}\right)^2} \times \frac{(x-y).2x-(x^2+y^2).1}{(x-y)^2}$$

$$= \frac{x^2-2xy-y^2}{(x^2+y^2)^2+(x-y)^2}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{x^2+2xy-y^2}{(x^2+y^2)^2+(x-y)^2}.$$

## PARTIAL DERIVATIVES OF HIGHER ORDER

NOTES

We can find partial derivatives of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ just as we found those of $u$, because $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ **and** $\frac{\partial \mathbf{u}}{\partial \mathbf{y}}$ **are themselves functions of x and y.**

The four derivatives thus obtained, called the **second order partial derivatives** of $u$, or $f(x, y)$ are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

and are denoted as

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} \quad \text{or} \quad f_{xx}, f_{yx}, f_{xy}, f_{yy}.$$

The partial derivatives $\frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x \partial y}$ are distinguished by the order in which $u$ is successively differentiated w.r.t. $x$ and $y$. But it will be seen that, in general, they are equal.

**Note.** $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right)$ **and** $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{y} \partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{y}}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)$.

The third and higher order partial derivatives are defined in the same manner.

**Caution.** $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x} \partial \mathbf{y}} \neq \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{y}}$.

### SOLVED EXAMPLES

**Example 1.** *Verify that* $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ *for the following functions :*

(*i*) $u = ax^2 + 2hxy + by^2$, (*ii*) $u = \tan^{-1}\left(\frac{x}{y}\right)$.

**Sol.** (*i*) $u = ax^2 + 2hxy + by^2$ ...(1)

Diff. (1) partially w.r.t. $x$, $\frac{\partial u}{\partial x} = 2ax + 2hy$

Now diff. partially w.r.t. $y$, $\frac{\partial^2 u}{\partial y \partial x} = 2h$ ...(2)

Again diff. (1) partially w.r.t. $y$,

$$\frac{\partial u}{\partial y} = 2hx + 2by$$

Now diff. partially w.r.t. $x$, $\frac{\partial^2 u}{\partial x \partial y} = 2h$ ...(3)

Hence from (2) and (3), $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

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(*ii*) $u = \tan^{-1}\left(\dfrac{x}{y}\right)$ ...(1)

Diff. (1) partially w.r.t. *x*,

$$\frac{\partial u}{\partial x} = \frac{1}{1+x^2/y^2} \times \frac{1}{y} = \frac{y}{x^2+y^2} \qquad \left| \because \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}\frac{\partial}{\partial x}(x) = \frac{1}{y}\right.$$

Now differentiating partially w.r.t. *y*,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \qquad ...(2)$$

Again diff. (1) partially w.r.t. *y*,

$$\frac{\partial u}{\partial y} = \frac{1}{1+x^2/y^2} \times \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2+y^2}$$

$$\left[\because \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{y.0 - x.1}{y^2} = -\frac{x}{y^2}\right]$$

Now differentiating partially w.r.t. *x*,

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{(x^2+y^2).1 - x.2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \qquad ...(3)$$

Hence from (2) and (3),

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

**Example 2.** *If* $z = \log\,(x^2+y^2) + \tan^{-1}\dfrac{y}{x}$*, prove that*

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

**Sol.** $z = \log\,(x^2+y^2) + \tan^{-1}\dfrac{y}{x}$ ...(1)

Diff. both sides of eqn. (1) partially w.r.t. *x*

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{x^2+y^2}(2x) + \frac{1}{1+\frac{y^2}{x^2}}\frac{(x.0-y.1)}{x^2}$$

$$= \frac{2x}{x^2+y^2} + \frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right) = \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2}$$

or $$\frac{\partial z}{\partial x} = \frac{2x-y}{x^2+y^2}$$

Again diff. partially w.r.t. *x*,

$$\frac{\partial^2 z}{\partial x^2} = \frac{(x^2+y^2)(2-0) - (2x-y)(2x+0)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2} = \frac{-2x^2+2xy+2y^2}{(x^2+y^2)^2} \qquad ...(2)$$

Now diff. both sides of eqn.(1) partially w.r.t. $y$,

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{1}{x^2+y^2}(2y) + \frac{x^2}{x^2+y^2}\frac{1}{x}(1) = \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}$$

Again diff. partially w.r.t. $y$,

$$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2+y^2)2-(2y+x)2y}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2-2xy}{(x^2+y^2)^2} = \frac{2x^2-2xy-2y^2}{(x^2+y^2)^2} \quad ...(3)$$

Adding equations (2) and (3), we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{-2x^2+2xy+2y^2}{(x^2+y^2)^2} + \frac{2x^2-2xy-2y^2}{(x^2+y^2)^2}$$

$$= \frac{-2x^2+2xy+2y^2+2x^2-2xy-2y^2}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = 0.$$

**Example 3.** *If* $\theta = t^n e^{-r^2/4t}$, *find the value of n which will make*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) \text{ equal to } \frac{\partial\theta}{\partial t}.$$

**Sol.** Here $\theta = t^n e^{-r^2/4t}$ ...(*i*)

(θ is an explicit function of $r$ and $t$)

Diff. both sides of (*i*) partially w.r.t. $t$,

$$\frac{\partial\theta}{\partial t} = nt^{(n-1)}e^{-r^2/4t} + t^n e^{-r^2/4t}\left(\frac{r^2}{4t^2}\right) \quad | \ u\,.\,v \text{ form}$$

$$= t^{n-2}\left(nt+\frac{r^2}{4}\right).\,e^{-r^2/4t} \quad ...(ii)$$

Again diff. both sides of (*i*) partially w.r.t. $r$,

$$\frac{\partial\theta}{\partial r} = t^n.e^{-r^2/4t}.\left(\frac{-2r}{4t}\right) = -\frac{1}{2}r.t^{n-1}.e^{-r^2/4t}$$

$$\therefore \quad r^2\frac{\partial\theta}{\partial r} = -\frac{1}{2}r^3.\,t^{n-1}e^{-r^2/4t}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{1}{2}r^3.t^{n-1}e^{-r^2/4t}\right)$$

$$= -\frac{1}{2}t^{n-1}\left[3r^2e^{-r^2/4t} + r^3\,.\,e^{-r^2/4t}\left(-\frac{2r}{4t}\right)\right]$$

$$= -\frac{1}{2}t^{n-1}\left[3r^2 - \frac{1}{2}.\frac{r^4}{t}\right]e^{-r^2/4t}$$

$$\therefore \quad \frac{1}{r^2}.\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2}t^{n-1}\left[3-\frac{1}{2}.\frac{r^2}{t}\right]e^{-r^2/4t} \quad ...(iii)$$

But $\frac{1}{r^2}.\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$ (given), therefore putting values from (*ii*) and (*iii*), we have

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$$-\frac{1}{2}t^{n-1}\left(3-\frac{1}{2}.\frac{r^2}{t}\right)e^{-r^2/4t} = t^{n-2}\left(nt+\frac{r^2}{4}\right)e^{-r^2/4t}$$

Dividing both sides by $t^{n-2}\, e^{-r^2/4t}$, we have

$$-\frac{1}{2}t\left(3-\frac{r^2}{2t}\right) = nt+\frac{r^2}{4} \quad \text{or} \quad -\frac{3}{2}t+\frac{r^2}{4} = nt+\frac{r^2}{4}$$

or $$-\frac{3}{2}t = nt \quad \therefore \quad n = -\frac{3}{2}.$$

**Example 4.** *If* $x = r\cos\theta,\ y = r\sin\theta$ *; prove that*

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right].$$

**Sol.** $\because \quad x = r\cos\theta,\ y = r\sin\theta$

[By looking at the answer we find that we need the partial derivatives of $r$ w.r.t. $x$ and $y$.

$\therefore$ Let us try to express $r$ as an *explicit* function of $x$ and $y$].

Squaring and adding $x = r\cos\theta,\ y = r\sin\theta$ ; we find that

$$r^2 = x^2+y^2 \qquad i.e., \qquad r = \sqrt{x^2+y^2}$$

$r$ is a symmetric function of $x$ and $y$.

Diff. partially w.r.t. $x$,

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2)^{-1/2}.2x = (x^2+y^2)^{-1/2}.x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \qquad \text{...(1)}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Again differentiating (1) partially w.r.t. $x$,

$$\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r\frac{\partial}{\partial x}(x)-x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \frac{r-x\frac{\partial r}{\partial x}}{r^2} = \frac{r-x.\frac{x}{r}}{r^2} \qquad \text{[By (1)]}$$

$$= \frac{r^2-x^2}{r^3} = \frac{y^2}{r^3} \qquad (\because\ r^2 = x^2+y^2)$$

Similarly, $$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$$

$$\text{L.H.S.} = \frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3}+\frac{x^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}$$

$$\text{R.H.S.} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right] = \frac{1}{r}\left(\frac{x^2+y^2}{r^2}\right) = \frac{1}{r}\left(\frac{r^2}{r^2}\right) = \frac{1}{r}$$

$\therefore$ L.H.S. = R.H.S.

**Example 5.** *Find the value of the parameter n so that*

*$V = r^n (3\cos^2\theta - 1)$ satisfies*

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

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**Sol.** $V = r^n (3\cos^2\theta - 1)$ ...(1) (given)

$\therefore \quad \frac{\partial V}{\partial r} = n\, r^{n-1}(3\cos^2\theta - 1)$

and $\quad \frac{\partial V}{\partial\theta} = r^n [6\cos\theta(-\sin\theta)] = -6\, r^n \cos\theta\sin\theta$

Putting these values of $\frac{\partial V}{\partial r}$ and $\frac{\partial V}{\partial\theta}$ in the given equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0, \text{ we have}$$

$$\frac{\partial}{\partial r}[r^2 . n\, r^{n-1}(3\cos^2\theta - 1)] + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}[-6\, r^n \cos\theta\sin\theta\sin\theta] = 0$$

*i.e.,* $\quad \frac{\partial}{\partial r}[n\, r^{n+1}(3\cos^2\theta - 1)] - 6\frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}(\cos\theta\sin^2\theta) = 0$

*i.e.,* $\quad \frac{\partial}{\partial r}[n\, r^{n+1}(3\cos^2\theta - 1)] - 6\frac{r^n}{\sin\theta}\frac{\partial}{\partial\theta}(\cos\theta - \cos^3\theta) = 0$

$[\because \cos\theta\sin^2\theta = \cos\theta(1 - \cos^2\theta) = \cos\theta - \cos^3\theta]$

*i.e.,* $\quad n(n+1)\, r^n(3\cos^2\theta - 1) - 6\frac{r^n}{\sin\theta}(-\sin\theta + 3\cos^2\theta\sin\theta) = 0$

*i.e.,* $\quad n(n+1)\, r^n(3\cos^2\theta - 1) - 6\frac{r^n}{\sin\theta}\sin\theta(3\cos^2\theta - 1) = 0$

or $\quad (3\cos^2\theta - 1)\, r^n\, [n(n+1) - 6] = 0$

But $\quad (V) = r^n(3\cos^2\theta - 1) \neq 0$ always [By (1)]

$\therefore \quad n(n+1) - 6 = 0$

or $\quad n^2 + n - 6 = 0$

or $\quad (n+3)(n-2) = 0$

$\therefore \quad n = -3, \quad n = 2.$

## EXERCISE A

**1.** Find the first order partial derivatives of

(*i*) $\log(x^2 + y^2)$ (*ii*) $\frac{1}{\sqrt{x^2 + y^2}}$ (*iii*) $\cos^{-1}\left(\frac{x}{y}\right)$

(*iv*) $\cos^{-1}\left(\frac{y}{x}\right)$ (*v*) $x^y + y^x$ (*vi*) $e^{x^y}$.

**2.** Find the second order partial derivatives of

(*i*) $\log(e^x + e^y)$ (*ii*) $e^{x-y}$.

**3.** Verify that $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x}$ where

(*i*) $z = ax^3 + 3bx^2y + 3cxy^2 + dy^3$ (*ii*) $z = \log(y\sin x + x\sin y)$

(*iii*) $z = \log\left(\frac{x^2 + y^2}{xy}\right)$ (*iv*) $z = \sin^{-1}\frac{x}{y}$.

**NOTES**

**4.** If $u = \log(\tan x + \tan y)$, prove that $\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2$.

**5.** Find the value of $\frac{\partial^2 z}{\partial x^2} - m^2 \frac{\partial^2 z}{\partial y^2}$ where $z = \tan(y + mx) + (y - mx)^{3/2}$.

**6.** (*a*) If $z(x + y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

$\left[\textbf{Hint.}\ z = \frac{x^2 + y^2}{x + y}.\right]$

(*b*) If $u = f(x^2 + y^2)$ ; prove that $\frac{\partial u}{\partial x} : \frac{\partial u}{\partial y} = x : y$

**7.** If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 r}{\partial x^2} \cdot \frac{\partial^2 r}{\partial y^2} = \left(\frac{\partial^2 r}{\partial x \partial y}\right)^2$.

**8.** If $x = r\cos\theta$, $y = r\sin\theta$ ; prove that $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$, except when $x = 0$, $y = 0$.

$\left[\textbf{Hint.}\ \text{Dividing } \tan\theta = \frac{y}{x} \quad \therefore \quad \theta = \tan^{-1}\frac{y}{x}.\right]$

**9.** If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

**10.** (*a*) If $u = e^x(x\cos y - y\sin y)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(*b*) If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$.

(*c*) If $z = \log(x^2 + y^2) + \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

(*d*) If $z = \cos(x + y) + \sin(x - y)$ ; prove that $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$.

**11.** If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

**12.** If $u = \log(x^3 + y^3 - x^2y - xy^2)$, prove that

(*i*) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x + y)^{-1}$

(*ii*) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -4(x + y)^{-2}$.

[**Hint.** $x^3 + y^3 - x^2y - xy^2 = (x + y)(x^2 + y^2 - xy) - xy(x + y)$
$= (x + y)(x^2 + y^2 - 2xy) = (x + y)(x - y)^2$.]

**13.** If $z = e^{ax + by} f(ax - by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

**14.** (*a*) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$.

NOTES

(*b*) If $u = \sqrt{x^2 + y^2 + z^2}$, show that

(*i*) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1.$

(*ii*) If $u = \sqrt{x^2 + y^2 + z^2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

### Answers

**1.** (*i*) $\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}$ (*ii*) $\frac{-x}{(x^2 + y^2)^{3/2}}, \frac{-y}{(x^2 + y^2)^{3/2}}$

(*iii*) $\frac{-1}{\sqrt{y^2 - x^2}}, \frac{x}{y\sqrt{y^2 - x^2}}$ (*iv*) $\frac{y}{x\sqrt{x^2 - y^2}}, \frac{-1}{\sqrt{x^2 - y^2}}$

(*v*) $yx^{y-1} + y^x \log y,\ x^y \log x + xy^{x-1}$ (*vi*) $e^{x^y} . yx^{y-1};\ e^{x^y} x^y \log x$

**2.** (*i*) $\frac{e^{x+y}}{(e^x + e^y)^2}, -\frac{e^{x+y}}{(e^x + e^y)^2}; \frac{e^{x+y}}{(e^x + e^y)^2}$ (*ii*) $e^{x-y}, -e^{x-y}, e^{x-y}$

**5.** 0.

## HOMOGENEOUS FUNCTIONS

**Definition 1.** In ordinary sense, $f(x, y)$ is said to be a *homogeneous function of order n*, if the **degree** of each of its terms in $x$ and $y$ is equal to $n$.

Thus,

$$p_0 x^n + p_1 x^{n-1} y + p_2 x^{n-2} y^2 + ..... + p_{n-1} xy^{n-1} + p_n y^n \quad ...(1)$$

is a homogeneous function in $x$ and $y$ of order $n$.

This definition of homogeneity applies to polynomial functions only. To widen the concept of homogeneity so as to bring even transcendental functions within its scope, we **define** $u$ as a *homogeneous function in x and y of order or degree n*, if it *can be expressed in the form*

$$\mathbf{x^n f\left(\frac{y}{x}\right)}$$

The above definition also covers the polynomial function (1), which can be written as

$$x^n\left[p_0 + p_1\frac{y}{x} + p_2\left(\frac{y}{x}\right)^2 + ...... + p_n\left(\frac{y}{x}\right)^n\right] = x^n f\left(\frac{y}{x}\right)$$

$\therefore$ It is a homogeneous function of order $n$.

But the functions $x^n \tan\frac{y}{x}, \frac{x+y}{\sqrt{x} - \sqrt{y}}$ are homogeneous according to the second definition only.

The degree of $x^n \tan\frac{y}{x}$ is $n$. $\quad \because\ x^n \tan\frac{y}{x} = x^n f\left(\frac{y}{x}\right)$

NOTES

and $$\frac{x+y}{\sqrt{x}-\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}\left(1-\sqrt{\frac{y}{x}}\right)} = x^{1/2}\cdot\left[\frac{1+y/x}{1-\sqrt{\frac{y}{x}}}\right]$$

$$= x^{1/2} f\left(\frac{y}{x}\right) \text{ so that it is of degree } \frac{1}{2}.$$

The function $\sin^{-1} y/x$ is a homogeneous function of degree 0, for it may be written as

$$x^0\left(\sin^{-1}\frac{y}{x}\right)$$

The function $\sin(x+y)$ is not a homogeneous function. It can be written in the form $\sin[x(1+y/x)]$, which is quite different from the form $x^n f(y/x)$.

**Note.** A homogeneous function in $x$ and $y$ of order $n$ can also be written as $\mathbf{y^n f\left(\frac{x}{y}\right)}$.

**Definition 2.** A function $u$ of **three** variables $x, y, z$ is said to be homogeneous function of degree $n$, if it can be expressed in the form

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right) \quad \text{or} \quad y^n \phi\left(\frac{x}{y}, \frac{z}{y}\right) \quad \text{or} \quad z^n \psi\left(\frac{x}{z}, \frac{y}{z}\right)$$

More generally, a function $u$ of several variables $x_1, x_2, x_3, \ldots\ldots, x_n$ is said to be homogeneous function of degree $m$ if it can be expressed in the form

$$u = x_1^m f\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots\ldots \frac{x_n}{x_1}\right) \quad \text{or} \quad x_2^m \phi\left(\frac{x_1}{x_2}, \frac{x_3}{x_2}, \ldots\ldots \frac{x_n}{x_2}\right) \text{ or etc.}$$

## IF u IS A HOMOGENEOUS FUNCTION OF x AND y OF DEGREE n, SHOW THAT $\frac{\partial u}{\partial x}$ AND $\frac{\partial u}{\partial y}$ ARE HOMOGENEOUS FUNCTIONS OF DEGREE (n – 1) EACH

**Proof.** Since $u$ is a homogeneous function of $x$ and $y$ of degree $n$, we can express $u$ in the form

$$u = x^n f\left(\frac{y}{x}\right).$$

$$\therefore \quad \frac{\partial u}{\partial x} = nx^{n-1}.f\left(\frac{y}{x}\right) + x^n.f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \qquad \left|\because \frac{d}{dx}(uv) = \frac{du}{dx}.v + u\frac{dv}{dx}\right.$$

$$= x^{n-1}\left[nf\left(\frac{y}{x}\right) + f'\left(\frac{y}{x}\right).\left(\frac{-y}{x}\right)\right]$$

$$= x^{n-1}\times\left(\text{a function of } \frac{y}{x}\right) = x^{n-1}\phi\left(\frac{y}{x}\right) \text{ (say)}$$

which is a homogeneous function in $x$ and $y$ of degree $(n-1)$.

Again $$\frac{\partial u}{\partial y} = x^n \cdot f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$= x^{n-1} \times \left(\text{a function of } \frac{y}{x}\right) = x^{n-1} \cdot \psi\left(\frac{y}{x}\right) \text{ (say)}$$

which is a homogeneous function in $x$ and $y$ of degree $(n - 1)$.

**Note.** The reader is suggested to observe from above that : $\partial u/\partial x$ is a function of both $x$ and $y$. Also $\partial u/\partial y$ is a function of both $x$ and $y$.

## EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

**If u be a homogeneous function of x and y of order n, then**

$$\mathbf{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = nu.}$$

**Proof.** Since $u$ is a homogeneous *function* of $x$ and $y$ of degree $n$, we may express it in the form,

$$u = x^n f\left(\frac{y}{x}\right)$$

$\therefore$ $$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) = nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)$$

Again $$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$$

$$\text{L.H.S.} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\left(n\,x^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$= n\,x^n f\left(\frac{y}{x}\right) - y.x^{n-1} f'\left(\frac{y}{x}\right) + yx^{n-1} f'\left(\frac{y}{x}\right)$$

$$= nx^n f\left(\frac{y}{x}\right) = nu = \text{R.H.S.}$$

**Note.** Euler's theorem can be extended to a homogeneous function of several variables. Thus, if $u$ be function of $m$ independent variables $x_1, x_2, x_3, \ldots\ldots, x_n$ of degree $n$, then this theorem states that

$$x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + \ldots\ldots + x_m \frac{\partial u}{\partial x_m} = nu.$$

The proof is similar to that for two variables.

For proof of Euler's theorem for homogeneous functions of three variables

## IF u IS A HOMOGENEOUS FUNCTION IN x AND y OF DEGREE n ; THEN PROVE THAT

$$\mathbf{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.}$$

**Proof.** $\because$ $u$ is a homogeneous function in $x$, $y$ of degree $n$

$\therefore$ By Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ ...(1) (Art. 8)

NOTES

Differentiating both sides of (1) partially w.r.t. $x$,

$$\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(y\frac{\partial u}{\partial y}\right)^{*}=\frac{\partial}{\partial x}(nu)$$

or
$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\,.\,1+y\,.\frac{\partial^2 u}{\partial x\partial y}=n\,.\frac{\partial u}{\partial x}$$

$\therefore$
$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial x} \qquad ...(2)$$

Again differentiating both sides of (1) partially w.r.t. $y$.
(This can be obtained (by interchanging $x$ and $y$ in (2)))

$$y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\partial y}=(n-1)\frac{\partial u}{\partial y} \qquad ...(3)$$

Multiplying (2) by $x$ ; (3) by $y$ and then adding ; we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=(n-1)\,nu$$

| Using (1)

$$=n(n-1)\,u.$$

## SOLVED EXAMPLES

**Example 6.** *Verify Euler's theorem for the function,*
$u=(x^{1/2}+y^{1/2})\,(x^n+y^n).$

**Sol.** $u=(x^{1/2}+y^{1/2})\,(x^n+y^n)$ ...(1)

$$=x^{1/2}\left(1+\sqrt{\frac{y}{x}}\right)x^n\left[1+\left(\frac{y}{x}\right)^n\right]=x^{n+1/2}\left(1+\sqrt{\frac{y}{x}}\right)\left[1+\left(\frac{y}{x}\right)^n\right]$$

It is a homogeneous function of order $n+\frac{1}{2}$.

$\therefore$ By Euler's theorem, we must have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\left(n+\frac{1}{2}\right)u \qquad ....(2)$$

We now proceed to verify (2).
From (1), we get

$$\frac{\partial u}{\partial x}=(x^{1/2}+y^{1/2})nx^{n-1}+(x^n+y^n)\frac{1}{2}x^{-1/2}$$

$\therefore$
$$x\frac{\partial u}{\partial x}=(x^{1/2}+y^{1/2})nx^n+\frac{1}{2}(x^n+y^n)x^{1/2} \qquad ...(3)$$

Similarly,
$$\frac{\partial u}{\partial y}=(x^{1/2}+y^{1/2})ny^{n-1}+(x^n+y^n)\,.\,\frac{1}{2}y^{-1/2}$$

$\therefore$
$$y\frac{\partial u}{\partial y}=(x^{1/2}+y^{1/2})ny^n+\frac{1}{2}(x^n+y^n)y^{1/2} \qquad ...(4)$$

Adding (3) and (4),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n(x^{1/2}+y^{1/2})(x^n+y^n)+\frac{1}{2}(x^n+y^n)(x^{1/2}+y^{1/2})$$

---

* Each of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ is a function of both $x$ and $y$.

$$= (x^{1/2} + y^{1/2})\,(x^n + y^n)\left[n + \frac{1}{2}\right] = \left(n + \frac{1}{2}\right)u.$$

Hence (2) is verified.

NOTES

**Example 7.** *If* $V = \cos^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, *show that*

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + \frac{1}{2}\cot V = 0.$$

**Sol.** $$V = \cos^{-1} \frac{x + y}{\sqrt{x} + \sqrt{y}}$$

$\therefore$ $$\cos V = \frac{x + y}{\sqrt{x} + \sqrt{y}} = \frac{x(1 + y/x)}{x^{1/2}\left(1 + \dfrac{y^{1/2}}{x^{1/2}}\right)} = x^{1/2}\left(\frac{1 + y/x}{1 + \sqrt{y/x}}\right)$$

$\therefore$ cos V is a homogeneous function of order $\frac{1}{2}$.

$\therefore$ $$x\frac{\partial}{\partial x}(\cos V) + y\frac{\partial}{\partial y}(\cos V) = \frac{1}{2}\cos V$$

or $$x(-\sin V)\frac{\partial V}{\partial x} + y(-\sin V)\frac{\partial V}{\partial y} = \frac{1}{2}\cos V$$

or $$-x\frac{\partial V}{\partial x} - y\frac{\partial V}{\partial y} = \frac{1}{2}\frac{\cos V}{\sin V}$$

$\therefore$ $$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + \frac{1}{2}\cot V = 0.$$

**Example 8.** *If* $u = \sin^{-1}\left(\dfrac{x}{y}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)$,

*show that* $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

**Sol.** $$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$= x^0\left[\sin^{-1}\left(\frac{1}{y/x}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right] = x^0\left[\text{a function of } \frac{y}{x}\right]$$

$\therefore$ $u$ is a homogeneous function of order 0

$\therefore$ By Euler's theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0 \times u = 0$.

**Example 9.** *If* $u = \dfrac{x^2y^2}{x + y}$, *show that* $y\dfrac{\partial^2 u}{\partial y^2} + x\dfrac{\partial^2 u}{\partial x \partial y} = 2\dfrac{\partial u}{\partial y}$.

**Sol.** $$u = \frac{x^2y^2}{x + y} = \frac{x^4\dfrac{y^2}{x^2}}{x\left(1 + \dfrac{y}{x}\right)} = x^3\frac{\left(\dfrac{y}{x}\right)^2}{1 + \dfrac{y}{x}} = x^3 f\left(\frac{y}{x}\right)$$

$u$ is a homogeneous function of degree 3 in $x$ and $y$.

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$\therefore$ By Euler's Theorem (Art. 8),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u.$$

Differentiating both sides partially w.r.t. $y$*,

$$\frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)=3\frac{\partial u}{\partial y} \quad \text{or} \quad x\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}.1=3\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}=2\frac{\partial u}{\partial y} \quad \text{or} \quad y\frac{\partial^2 u}{\partial y^2}+x\frac{\partial^2 u}{\partial x\partial y}=2\frac{\partial u}{\partial y}$$

which is the result to be proved.

## EXERCISE B

**1.** (*a*) If $u=x^m f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=mu$.

(*b*) If $z = F(x, y)$ be a homogeneous functions of $x$, $y$ of degree $m$, prove that

$$x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}=mz$$

Deduce the result if $m = 1$.

[**Hint.** For (*a*) and (*b*) : It is Art. 8]

(*c*) If $z = F(x, y)$ be a homogeneous function of $x$, $y$ of degree $m$, prove that

$$x^2\frac{\partial^2 F}{\partial x^2}+2xy\frac{\partial^2 F}{\partial x\partial y}+y^2\frac{\partial^2 F}{\partial y^2}=m(m-1)z.$$

[**Hint.** It is Art. 9]

**2.** Verify Euler's Theorem for the following functions :

(*i*) $\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$ (*ii*) $\frac{1}{x^2+xy+y^2}$ (*iii*) $x^n \sin\frac{y}{x}$ (*iv*) $x^4\log\frac{y}{x}$.

**3.** (*i*) If $u=xf\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$

(*ii*) If $u=f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$.

[**Hint.** $u=f\left(\frac{y}{x}\right)=x^{\circ} f\left(\frac{y}{x}\right)$ is a homogeneous function of degree 0 in $x$ and $y$.]

**4.** If $u=xy\,f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u.$

**5.** If $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show that $\frac{\partial u}{\partial x}=-\frac{y}{x}\frac{\partial u}{\partial y}$.

**6.** (*a*) If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u.$

(*b*) If $\sin u=\frac{x^2y^2}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\tan u.$

*This is being done because we find $\frac{\partial^2 u}{\partial y^2}$ in the result to be proved.

**7.** (*a*) If $u = \tan^{-1} \dfrac{x^3 + y^3}{x - y}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u$.

(*b*) If $u = \tan^{-1} \dfrac{x^2 + y^2}{x + y}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{1}{2} \sin 2u$.

(*c*) If $u = \tan^{-1} \left( \dfrac{x^2 - y^2}{x + y} \right)$, then prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{1}{2} \sin 2u$.

(*d*) If $u = \tan^{-1} \left( \dfrac{x + y}{\sqrt{x} + \sqrt{y}} \right)$, then prove that $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = \dfrac{1}{4} \sin 2u$.

(*e*) If $u = \sec^{-1} \left( \dfrac{x^3 + y^3}{x + y} \right)$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 2 \cot u$.

**8.** (*a*) If $u = \log \dfrac{x^4 + y^4}{x - y}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 3$.

[**Hint.** $\dfrac{x^4 + y^4}{x - y} = e^u = z$ is a homogeneous function of degree 3.]

(*b*) If $z = \log \dfrac{x^2 + y^2}{x + y}$, prove that $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 1$.

(*c*) If $u = \log (\sqrt{x} + \sqrt{y})$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{1}{2}$.

(*d*) If $u = \log \dfrac{x^4 + y^4 + x^2 y^2}{x + y + \sqrt{xy}}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 3$.

**9.** (*a*) If $\sin v = \dfrac{x + 2y + 3z}{\sqrt{x^8 + y^8 + z^8}}$, show that $x \dfrac{\partial v}{\partial x} + y \dfrac{\partial v}{\partial y} + z \dfrac{\partial v}{\partial z} + 3 \tan v = 0$.

(*b*) If $u = (x^2 + y^2 + z^2)^{-1/2}$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = -u$.

[**Hint.** See Note Art. 8.]

**10.** If $u = \dfrac{xy}{x + y}$, show that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$.

**11.** If $u = x\phi \left( \dfrac{y}{x} \right) + \psi \left( \dfrac{y}{x} \right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0$.

[**Hint.** Write $u = v + w$, where $v = x\phi \left( \dfrac{y}{x} \right)$ and $w = \psi \left( \dfrac{y}{x} \right)$, $v$ and $w$ are homogeneous functions of degree 1 and 0 respectively.]

**12.** If $u = \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \,.\, \cos 2u}{4 \cos^3 u}.$$

**13.** (*a*) If $u = \sin^{-1} \left[ \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} \right]^{1/2}$, then show that

(*i*) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{-1}{12} \tan u$.

(*ii*) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\tan u}{144} (13 + \tan^2 u)$

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(*b*) If $u = \text{cosec}^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u).$$

[**Hint.** $\text{cosec}^{-1}t = \sin^{-1}\dfrac{1}{t}$ ]

(*c*) If $\sin^2 u = \left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u).$$

**14.** If $u = \sin^{-1}(x^2 + y^2)^{1/5}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{2}{25}\tan u\,(2\tan^2 u - 3).$$

**15.** If $f(x, y) = (x^2 + y^2)^{1/3}$, prove that $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = -\dfrac{2}{9}f$.

**16.** (*a*) If $u = \dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y \partial x} = 2\dfrac{\partial u}{\partial x}$.

(*b*) If $u = \dfrac{xy}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{1}{2}\dfrac{\partial u}{\partial x}$.

## COMPOSITE FUNCTIONS

*If u is given to be a function of the variables x, y and these variables themselves are given to be the functions of the variable t, then u is said to be a composite function of the variable t. Thus the relations*

$$u = f(x, y)\ ;\ x = \phi(t),\ y = \psi(t)$$

*define u as a composite function of t.*

Again, the relations

$$z = f(x, y),\ x = \phi(u, v),\ y = \psi(u, v)$$

define $z$ as a composite function of $u$ and $v$.

## DIFFERENTIATION OF COMPOSITE FUNCTIONS

*If u is a composite function of t, defined by the relations,*

$$u = f(x, y)\ ;\ x = \phi(t),\ y = \psi(t)$$

*where u possesses continuous first order partial derivatives w.r.t. x and y ; and x and y possess continuous derivatives w.r.t. 't' then*

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

**Proof.** Here $u = f(x, y)$ ...(*i*)

Let $t$ receive an increment $\delta t$ and let the corresponding increment in $x, y, u$ by $\delta x, \delta y, \delta u$ respectively. Then, we have

$$u + \delta u = f(x + \delta x, y + \delta y) \quad \text{... (ii)}$$

Subtracting (*i*) from (*ii*), we get

$$\delta u = f(x + \delta x, y + \delta y) - f(x, y)$$
$$= [f( x + \delta x, y + \delta y) - f( x, y + \delta y)] + [f(x, y + \delta y) - f( x, y)]$$

[By subtracting and adding $f(x, y + \delta y)$]

Applying Lagrange's Mean Value Theorem

[*i.e.*, $f(a + h) = f(a) + hf'(a + \theta h)$ ; $0 < \theta < 1$] to the two differences on the right, we have

$$\delta u = \delta x f_x(x + \theta_1 \delta x, y + \delta y) + \delta y \,.\, f_y(x, y + \theta_2 \delta y) \ ; \ [0 < \theta_1 < 1 \text{ and } 0 < \theta_2 < 1]$$

Dividing both sides by $\delta t$, $\dfrac{\delta u}{\delta t} = \dfrac{\delta x}{\delta t};\ f_x(x + \theta_1 \delta x, y + \delta y) + \dfrac{\delta y}{\delta t} \,.\, f_y(x, y + \theta_2\, \delta y)$

...(*iii*)

Let $\delta t \to 0$, so that $\delta x$ and $\delta y \to 0$.

Because of the continuity of partial derivatives, we get

$$\underset{\delta x,\, \delta y \to 0}{\text{Lt}} f_x(x + \theta_1\, \delta x,\, y + \delta y) = f_x(x, y) = \frac{\partial u}{\partial x}$$

and

$$\underset{\delta y \to 0}{\text{Lt}} f_y(x,\, y + \theta_2 \delta y) = f_y(x, y) = \frac{\partial u}{\partial y}$$

Hence in the limit, (*iii*) becomes

$$\mathbf{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}} \quad \text{...(iv)}$$

**Note 1.** $\dfrac{du}{dt}$ is called the *total derivative u*, w.r.t. *t*.

**2.** $\dfrac{du}{dt}$ can also be replaced by $\dfrac{df}{dt}$.

**Cor. 1. An important special case.** By supposing $t$ to be the same as $x$ in the above article, we get the following theorem :

When $u$ is a function of $x$ and $y$, and $y$ is a function of $x$, then the total differential co-efficient of $u$ w.r.t. $x$ is given by

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \text{or} \quad \mathbf{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}}.$$

**Cor 2.** Let $z = f(x, y)$, where
$x = \phi(u, v),\ y = \psi(u, v)$.

Since $z$ is a composite function of two variables $u$ and $v$, we may find $\partial z/\partial u$ and $\partial z/\partial v$.

To obtain $\partial z/\partial u$, we regard $v$ as a constant, so that $x$ and $y$ may be supposed to be functions of $u$ only. Then by the equation (*iv*) of above theorem, we get

$$\mathbf{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}} \quad \text{...(v)}$$

Similarly,

$$\mathbf{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}} \quad \text{...(vi)}$$

The ordinary (*i.e.,* total) derivatives of equation (*iv*) above have been replaced by partial derivatives because $x$, $y$ are functions of two variables $u$ and $v$.

**Remark.** The result of the above Cor. 2 is true even when $z$ is a composite function of three variables $u$, $v$, $w$.



## CHANGE OF VARIABLES

If $\qquad z = f(x, y) \qquad$ ...(1)

where $\qquad x = \phi(u, v)$ and $y = \psi(u, v) \qquad$ ...(2)

then by Art. 11 it is possible to change expressions involving $z$, **x, y,** $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ etc. to expressions involving $z$, **u, v,** $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$ etc.

The necessary formulae for change of these variables are given by equations (*v*) and eqn. (*vi*) of Cor. 2 Art. 11.

**Let us treat z as a composite function of u and v.**

If $v$ is regarded as constant ; then $x$, $y$, $z$ be functions of $u$ alone.

∴ By eqn. (*v*) of Art. 11,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad \text{...(3)}$$

Similarly regarding $u$ as constant, $x$, $y$, $z$ will be functions of $v$ alone.

∴ By eqn. (*vi*) of Art. 11,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \qquad \text{...(4)}$$

On solving equations (3) and (4) as simultaneous equations in $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we get their values in terms of $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$, $u$ and $v$.

If instead of equations (2), $u$ and $v$ are given in terms of $x$ and $y$ say

$$u = \varepsilon(x, y) \text{ and } v = \eta(x, y) \qquad \text{...(5)}$$

then it is easier to use the formulae (*treating z as a composite function of x and y*)

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \text{...(6)}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad \text{...(7)}$$

The higher derivatives of $z$ can be obtained by repeated application of the formulae (3) and (4) or by repeated application of the formulae (6) and (7).

**Remark.** If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$ ; then the equation (*iv*) of Art. 11 namely $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ changes the variables $u$, $x$ and $y$ in terms of $u$ and $t$.

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## SOLVED EXAMPLES

**Example 10.** *Find* $\frac{du}{dt}$ *when* $u = x^2 + y^2$, $x = at^2$ ; $y = 2at$, *verify by direct substitution.*

**Sol.** Now $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ (By eqn. (*iv*) of Art. 11)

$$= 2x \,.\, 2at + 2y \,.\, 2a = 2.at^2 \,.\, 2at + 2.2at.2a$$

$$= 4a^2t^3 + 8a^2t = 4a^2t(t^2 + 2)$$

Again, $u = x^2 + y^2 = a^2t^4 + 4a^2t^2$

$$\therefore \quad \frac{du}{dt} = 4a^2t^3 + 8a^2t = 4a^2t(t^2 + 2)$$

Hence the verification.

**Example 11.** *If* $z = \sqrt{x^2 + y^2}$ *and* $x^3 + y^3 + 3axy = 5a^2$, *find the value of dz/dx when* $x = a$, $y = a$.

**Sol.** The relation $x^3 + y^3 + 3axy - 5a^2 = 0$ defines $y$ as a function of $x$, therefore, $z$ is composite function of $x$.

$$\therefore \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad | \text{ By Cor. 1, Art. 11}$$

$$= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \qquad \ldots(1)$$

From $z = \sqrt{x^2 + y^2}$, $\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2} \,.\, 2x = \frac{x}{\sqrt{x^2 + y^2}}$

and $\frac{\partial z}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2} \,.\, 2y = \frac{y}{\sqrt{x^2 + y^2}}$

Also differentiating the given relation w.r.t. $x$, we get

$$3x^2 + 3y^2 \,.\, \frac{dy}{dx} + 3a\left(x\frac{dy}{dx} + y\right) = 0$$

or $(3y^2 + 3ax)\frac{dy}{dx} = -(3x^2 + 3ay)$ or $\frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}$.

Substituting in (1), we have

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{x^2 + ay}{y^2 + ax}\right)$$

$$\therefore \quad \left[\frac{dz}{dx}\right]_{\substack{x = a \\ y = a}} = \frac{a}{\sqrt{a^2 + a^2}} + \frac{a}{\sqrt{a^2 + a^2}}\left(-\frac{a^2 + a^2}{a^2 + a^2}\right)$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0.$$

**Example 12.** *If z is a function of x and y ; where* $x = e^u + e^{-v}$ *and* $y = e^{-u} - e^v$, *show that*

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

**Sol.** Here $z$ is a composite function of $u$ and $v$.

($\because$ $z$ is a function of both $x$ and $y$ and $x$, $y$ are functions of $u$, $v$).

$\therefore$ By eqns. ($v$) and ($vi$) of Cor. 2, Art. 11 ;

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$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad ...(1)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad ...(2)$$

But $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$

$\therefore$ $\dfrac{\partial x}{\partial u} = e^u$ and $\dfrac{\partial x}{\partial v} = -e^{-v}$

Also $\dfrac{\partial y}{\partial u} = -e^{-u}$ and $\dfrac{\partial y}{\partial v} = -e^v$

Putting these values in (1) and (2), we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} e^u + \frac{\partial z}{\partial y}(-e^{-u}) = \frac{\partial z}{\partial x} . e^u - \frac{\partial z}{\partial y} . e^{-u} \qquad ...(3)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^v) = -\frac{\partial z}{\partial x} . e^{-v} - \frac{\partial z}{\partial y} e^v \qquad ...(4)$$

Subtracting (3) and (4), we get

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^v) = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} .$$

**Example 13.** *Prove that* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2}$

*where* $x = u \cos \alpha - v \sin \alpha$, $y = u \sin \alpha + v \cos \alpha$.

*Or*

*By changing the independent variables u and v to x and y by means of the relations* $x = u \cos \alpha - v \sin \alpha$, $y = u \sin \alpha + v \cos \alpha$ *; show that*

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \text{ transforms into } \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} .$$

**Sol.** Let us treat $z$ as a composite function of $u$ and $v$.

$\therefore$
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad ...(1)$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \qquad ...(2)$$

(By equations (3) and (4) of Art. 12)

But $x = u \cos \alpha - v \sin \alpha$ and $y = u \sin \alpha + v \cos \alpha$

$\therefore$ $\dfrac{\partial x}{\partial u} = \cos \alpha$ and $\dfrac{\partial x}{\partial v} = -\sin \alpha$

Also $\dfrac{\partial y}{\partial u} = \sin \alpha$ and $\dfrac{\partial y}{\partial v} = \cos \alpha$

Putting these values in (1) and (2), we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha \qquad ...(3)$$

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and $$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} \sin \alpha + \frac{\partial z}{\partial y} \cos \alpha \qquad \text{...(4)}$$

or $$\frac{\partial}{\partial u}(z) = \left(\cos\alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}\right) z$$

$\Rightarrow$ $$\frac{\partial}{\partial u} = \cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial y} \qquad \text{...(5)}$$

and $$\frac{\partial}{\partial v}(z) = \left(-\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y}\right) z$$

$\Rightarrow$ $$\frac{\partial}{\partial v} = -\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y} \qquad \text{...(6)}$$

Now $$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(\cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha\right)$$

[by (5)] [By (3)]

$$= \cos^2 \alpha \frac{\partial^2 z}{\partial x^2} + \sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 z}{\partial y^2}$$

or $$\frac{\partial^2 z}{\partial u^2} = \cos^2 \alpha \frac{\partial^2 z}{\partial x^2} + 2\sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \alpha \frac{\partial^2 z}{\partial y^2} \qquad \text{...(7)}$$

Again $$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = \left(-\sin\alpha \frac{\partial}{\partial x} + \cos\alpha \frac{\partial}{\partial y}\right)\left(-\frac{\partial z}{\partial x}\sin\alpha + \frac{\partial z}{\partial y}\cos\alpha\right)$$

[By (6)] [by(4)]

$$= \sin^2 \alpha \frac{\partial^2 z}{\partial x^2} - \sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} - \sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2 \alpha \frac{\partial^2 z}{\partial y^2}$$

or $$\frac{\partial^2 z}{\partial v^2} = \sin^2 \alpha \frac{\partial^2 z}{\partial x^2} - 2\sin\alpha \cos\alpha \frac{\partial^2 z}{\partial x \partial y} + \cos^2 \alpha \frac{\partial^2 z}{\partial y^2} \qquad \text{...(8)}$$

Adding eqns. (7) and (8), we have

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (\cos^2 \alpha + \sin^2 \alpha)\frac{\partial^2 z}{\partial x^2} + (\sin^2 \alpha + \cos^2 \alpha)\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

## IMPLICIT RELATION OF x AND y

In first year calculus, we were mainly concerned with the case in which $y$ is expressed *explicitly i.e.*, directly in terms of $x$. Cases however are of frequent occurrence in which $y$ is not expressed directly in terms of $x$, but its functionality is implied by an algebraic relation $f(x, y) = 0$ connecting $x$ and $y$.

Such relations **f(x, y) = c** (where $y$ is not explicitly in terms of $x$) are called **Implicit** functions and such equations define $y$ as an implicit function of $x$.

## DIFFERENTIATION OF IMPLICIT EQUATIONS

NOTES

**To find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for an implicit equation f(x, y) = 0 or f(x, y) = c.**

(*a*) **To find $\frac{dy}{dx}$ from the equation f(x, y) = 0 or f(x, y) = c**

Now $f(x, y)$ is a function of two variables $x$, $y$ and $y$ itself is a function of $x$, so that we may consider $f(x, y)$ as a composite function of $x$. Its derivative w.r.t. $x$ is

$$\frac{df}{dx} = \frac{\partial f}{\partial x}.\frac{dx}{dx} + \frac{\partial f}{\partial y}.\frac{dy}{dx} \quad | \text{ Cor. 1, Art. 11}$$

or
$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} \quad ...(1)$$

But
$$\frac{df}{dx} = 0 \quad | \because \ f(x, y) = c$$

Hence
$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} = 0$$

$\therefore$
$$\mathbf{\frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}} = -\frac{f_x}{f_y}, \text{ if } f_y \neq 0.$$

**Remark.** The reader is suggested to proceed as in Art. 11, if the reader is interested in an **independent** proof.

(*b*) **If f(x, y) = 0 and $f_y \neq 0$, find $\frac{d^2y}{dx^2}$.**

We have proved above in part (*a*) that

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

or
$$\frac{dy}{dx} = -\frac{F}{G}, \quad \text{where } F = \frac{\partial f}{\partial x} \text{ and } G = \frac{\partial f}{\partial y}$$

are **implicit** functions of $x$ and $y$.

Differentiating again w.r.t. $x$, we have

$$\frac{d^2y}{dx^2} = -\frac{\left[G\frac{dF}{dx} - F\frac{dG}{dx}\right]}{G^2}.$$

But replacing $f$ by F and G in equation (1) of part (*a*), we have

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \text{ and } \frac{dG}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx}.$$

$\therefore$
$$\frac{d^2y}{dx^2} = -\frac{\left[G\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}\right) - F\left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx}\right)\right]}{G^2}$$

Putting $\mathbf{F} = \frac{\partial f}{\partial x}$ , $\mathbf{G} = \frac{\partial f}{\partial y}$ and $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$ (from part ($a$)) ; we have

$$\frac{d^2y}{dx^2} = -\frac{\left[\frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y}\left(\frac{-\partial f/\partial x}{\partial f/\partial y}\right)\right) - \frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\left(-\frac{\partial f/\partial x}{\partial f/\partial y}\right)\right)\right]}{\left(\frac{\partial f}{\partial y}\right)^2}$$

or

$$\frac{d^2y}{dx^2} = -\frac{\left[\frac{\partial f}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x \partial y}\right) - \frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial y^2}\frac{\partial f}{\partial x}\right)\right]}{\left(\frac{\partial f}{\partial y}\right)^3}$$

or

$$\mathbf{\frac{d^2y}{dx^2} = -\frac{\left[\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2\right]}{\left(\frac{\partial f}{\partial y}\right)^3}}$$

## SOLVED EXAMPLES

**Example 14.** *If $x^3 + y^3 - 3axy = 0$, prove that* $\frac{d^2y}{dx^2} = \frac{-2a^3xy}{(y^2 - ax)^3}$.

*Also find the value of* $\frac{d^2y}{dx^2}$ *at the point* $(\frac{3}{2}a, \frac{3}{2}a)$.

**Sol.** Here $f(x, y) = x^3 + y^3 - 3axy = 0$ ...(1)

$\therefore$ $\frac{\partial f}{\partial x} = 3x^2 - 3ay \,;\, \frac{\partial f}{\partial y} = 3y^2 - 3ax$

$\frac{\partial^2 f}{\partial x^2} = 6x \,;\, \frac{\partial^2 f}{\partial x \partial y} = -3a \,;\, \frac{\partial^2 f}{\partial y^2} = 6y$

$\therefore$ $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}$ .

Also $$\frac{d^2y}{dx^2} = -\frac{\left[\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2\right]}{\left(\frac{\partial f}{\partial y}\right)^3}$$

$$= -\frac{[6x(3y^2 - 3ax)^2 - 2(3x^2 - 3ay)(3y^2 - 3ax)(-3a) + 6y(3x^2 - 3ay)^2]}{(3y^2 - 3ax)^3}$$

$$= -\frac{54}{27(y^2 - ax)^3}[x(y^2 - ax)^2 + a(x^2 - ay)(y^2 - ax) + y(x^2 - ay)^2]$$

$$= -\frac{2}{(y^2 - ax)^3}\,[xy^4 - 2ax^2y^2 + a^2x^3 + ax^2y^2 - a^2x^3 - a^2y^3$$
$$+ a^3xy + yx^4 - 2ax^2y^2 + a^2y^3]$$

NOTES

$$= -\frac{2}{(y^2 - ax)^3}\,[xy^4 + x^4y - 3ax^2y^2 + a^3xy]$$

$$= -\frac{2}{(y^2 - ax)^3}\,[xy(x^3 + y^3 - 3axy) + a^3xy]$$

$$= -\frac{2a^3xy}{(y^2 - ax)^3} \qquad \Big|\ \because\ x^3 + y^3 - 3axy = 0 \text{ (given)}$$

$\therefore$ At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, the value of $\frac{d^2y}{dx^2} = \dfrac{2a^3 \cdot \frac{3a}{2} \cdot \frac{3a}{2}}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^3} = -\frac{32}{3a}.$

**Example 15.** *If* $f(x, y, z) = c$, *find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$.

**Sol.** $f(x, y, z) = c$ ...(1)

This relation defines $z$ as a function of $x$ and $y$. In order to find $\frac{\partial z}{\partial x}$, we **regard y as constant in (1)** and

Therefore, $$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \qquad \Big|\ \because\ \text{By Art. 14}(a),\ \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Similarly, $$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

**Note.** We have the same results when $f(x, y, z) = 0$.

**Example 16.** *If* $x = u^2 - v^2$, $y = 2uv$ ; *find* $\frac{\partial u}{\partial x}$*, $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

**Sol.** The given equations are

$$x = u^2 - v^2 \quad ...(1) \qquad y = 2uv \quad ...(2)$$

Differentiating both the equations (1) and (2) partially w.r.t. $x$ ; (treating $y$ as a constant)

$$1 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} \quad ...(3) \qquad \text{and} \qquad 0 = 2\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}\right) \quad ...(4)$$

(By Product Rule)

Let us solve (3) and (4) for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$.

From (4), $2 \neq 0$,

$$\therefore \quad u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0 \quad \text{or} \quad u\frac{\partial v}{\partial x} = -v\frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial v}{\partial x} = -\frac{v}{u}\frac{\partial u}{\partial x} \quad ...(5)$$

*We require partial derivatives of $u$ and $v$ w.r.t. $x$ and $y$ whereas $x$ and $y$ are given to be functions of $u$ and $v$.

$\therefore$ $x$ and $y$ are to be treated as independent variables.

Putting this value of $\frac{\partial v}{\partial x}$ from (5) in (3), we have

$$1 = 2u\frac{\partial u}{\partial x} + \frac{2v^2}{u}\frac{\partial u}{\partial x}$$

**NOTES**

$$\therefore \quad u = 2u^2\frac{\partial u}{\partial x} + 2v^2\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial x}(u^2 + v^2)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{u}{2(u^2 + v^2)}$$

$$\therefore \text{ From (5),} \quad \frac{\partial v}{\partial x} = -\frac{v}{u} \times \frac{u}{2(u^2 + v^2)} = -\frac{v}{2(u^2 + v^2)}$$

Similarly, differentiating (1) and (2) partially w.r.t. $y$ and solving the resulting equations for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ ; we have

$$\frac{\partial u}{\partial y} = \frac{v}{2(u^2 + v^2)} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{u}{2(u^2 + v^2)}.$$

## EXERCISE C

**1.** (*a*) Find $\frac{du}{dt}$, when $u = xy^2 + x^2y$, $x = at^2$, $y = 2at$

Verify by direct substitution.

(*b*) If $z = \tan^{-1}\frac{y}{x}$, $x = \log t$, $y = e^t$; find $\frac{dz}{dt}$.

**2.** Find the total derivative of $u$ with respect to $t$, when

(*a*) $u = \cosh\left(\frac{y}{x}\right)$, where $x = t^2$, $y = e^t$.

(*b*) $u = e^x \sin y$ where $x = \log t$, $y = t^2$.

Also verify by direct calculations.

**3.** If $z = u^2 + v^2$, $u = r\cos\theta$, $v = r\sin\theta$ ; find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$.

**4.** If $x = u + v$, $y = uv$ and $z$ is a function of $x, y$ ; show that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}.$$

**5.** If $u = f(r, s)$ ; $r = x + y$, $s = x - y$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

**6.** If $z = f(x, y)$ where $x = e^u \cos v$ and $y = e^u \sin v$, show that $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$.

**7.** If $z = u^2 + v^2 + w^2$, where $u = ye^x$, $v = xe^{-y}$, $w = \frac{y}{x}$ ; find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

**8.** If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ ; show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

**NOTES**

**9.** Find the differential co-efficient of $x^2y$ w.r.t. $x$ when $x$ and $y$ are connected by the relation $x^2 + xy + y^2 = 1$.

**10.** Find $\frac{du}{dx}$ if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$.

**11.** By changing the independent variables $x$ and $y$ to $u$ and $v$ by means of the relations $u = x - ay$, $v = x + ay$, show that $a^2 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}$ transforms into $4a^2 \frac{\partial^2 z}{\partial u\, \partial v}$.

**12.** If $z$ is a function of $x$ and $y$ and $u$ and $v$ be two other variables such that $u = lx + my$, $v = ly - mx$. Show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$.

**13.** If $x = e^r \cos\theta$, $y = e^r \sin\theta$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right)$.

**14.** If $x = \rho\cos\phi$, $y = \rho\sin\phi$ ; show that

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial v}{\partial \phi}\right)^2.$$

**15.** Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2}$

where $x = s\cos\alpha - t\sin\alpha$ and $y = s\sin\alpha + t\cos\alpha$.

**16.** (*a*) Show that $\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$

where $f(x, y) = 0$ and $\phi(z, y) = 0$ are given functional equations.

(*b*) Show that $\frac{\partial \phi}{\partial v}\frac{\partial f}{\partial w}\frac{dw}{du} = \frac{\partial \phi}{\partial u}\frac{\partial f}{\partial v}$

where $\phi(u, v) = 0$ and $f(w, v) = 0$ are given functional equations.

**17.** If $\phi(x, y, z) = 0$, show that $\left(\frac{\partial y}{\partial z}\right)_{x\text{ const}}\left(\frac{\partial z}{\partial x}\right)_{y\text{ const}}\left(\frac{\partial x}{\partial y}\right)_{z\text{ const}} = -1$.

**18.** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from the following implicit relations :

(*i*) $x^2 + y^2 = a^2$  (*ii*) $x^{2/3} + y^{2/3} = a^{2/3}$

(*iii*) $x^5 + y^5 - 5a^3xy = 0$

**19.** If $y^3 - 3ax^2 + x^3 = 0$, then $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$.

**20.** (*a*) If A, B, C are the angles of a triangle such that

$\sin^2 A + \sin^2 B + \sin^2 C$ = constant, prove that $\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}$.

(*b*) If $\alpha, \beta, \gamma$ are the angles of a triangle such that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = k$ (a constant), prove that $\frac{d\alpha}{d\beta} = \frac{\tan\gamma - \tan\beta}{\tan\alpha - \tan\gamma}$.

**21.** If $x = u^2 - v$, $y = v^2 - u$ ; find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

**22.** If U is a homogeneous function of $x$, $y$, $z$ of order $n$, prove that

$$x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}+z\frac{\partial U}{\partial z}=nU.$$

**Sol.** This is Euler's theorem for a homogeneous function of three independent variables.

NOTES

Here by Def. 2, Art. 7,

$$U=x^n f\left(\frac{y}{x},\frac{z}{x}\right)=x^n f(u,v), \text{ where } \frac{y}{x}=u, \frac{z}{x}=v$$

Now $f(u, v)$ is a composite function of $x$, $y$, $z$.

$$\therefore \quad \frac{\partial U}{\partial x}=nx^{n-1}f(u,v)+x^n\frac{\partial f}{\partial x}=nx^{n-1}f(u,v)+x^n\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$

But $$\frac{\partial u}{\partial x}=-\frac{y}{x^2}, \frac{\partial v}{\partial x}=-\frac{z}{x^2}$$

Hence, $$\frac{\partial U}{\partial x}=nx^{n-1}f(u,v)-x^{n-2}\left(y\frac{\partial f}{\partial u}+z\frac{\partial f}{\partial v}\right)$$

Again, $$\frac{\partial U}{\partial y}=x^n\frac{\partial f}{\partial y}=x^n\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\right)=x^{n-1}\left(\frac{\partial f}{\partial u}\right), \text{ because } \frac{\partial u}{\partial y}=\frac{1}{x}, \frac{\partial v}{\partial y}=0$$

Similarly, $$\frac{\partial U}{\partial z}=x^{n-1}\frac{\partial f}{\partial v}$$

$$\therefore \quad x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}+z\frac{\partial U}{\partial z}$$

$$=x\left[nx^{n-1}f(u,v)-x^{n-2}y\frac{\partial f}{\partial u}-x^{n-2}z\frac{\partial f}{\partial v}\right]+yx^{n-1}\frac{\partial f}{\partial u}+zx^{n-1}\frac{\partial f}{\partial v}$$

$$=nx^n f(u,v)=nU.$$

**23.** If $f(x, y) = 0$, prove that $\dfrac{\partial^2 f}{\partial x^2}+2\dfrac{\partial^2 f}{\partial x\partial y}\dfrac{dy}{dx}+\dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{dy}{dx}\right)^2+\dfrac{\partial f}{\partial y}\cdot\dfrac{d^2y}{dx^2}=0$,

[**Hint.** It is Art. 14 ($b$).]

## Answers

**1.** ($a$) $\dfrac{du}{dt}=2a^3t^3(8+5t)$ ($b$) $\dfrac{e^t(t\log t-1)}{t[(\log t)^2+e^{2t}]}$

**2.** ($a$) $\dfrac{du}{dt}=\dfrac{1}{x^2}(xe^t-2yt)\sinh\dfrac{y}{x}$ ($b$) $\dfrac{du}{dt}=\dfrac{e^x}{t}(\sin y+2t^2\cos y)$ where $x=\log t$, $y=t^2$

**3.** $2r$, 0 **7.** $2y^2e^{2x}+2xe^{-2y}-\dfrac{2y^2}{x^3}$, $2ye^{2x}-2x^2e^{-2y}+\dfrac{2y}{x^2}$

**9.** $\dfrac{x(4y^2+xy-2x^2)}{x+2y}$ **10.** $2x\cdot\cos(x^2+y^2)\left[1-\dfrac{a^2}{b^2}\right]$

**18.** ($i$) $-\dfrac{x}{y}, -\dfrac{a^2}{y^3}$ ($ii$) $\dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}, \dfrac{d^2y}{dx^2}=\dfrac{a^{2/3}}{3x^{4/3}y^{1/3}}$

($iii$) $-\dfrac{(x^4-a^3y)}{(y^4-a^3x)}, -\dfrac{6a^3xy(x^3y^3+2a^6)}{(y^4-a^3x)^3}$ **21.** $\dfrac{\partial u}{\partial x}=\dfrac{2v}{4uv-1}$ and $\dfrac{\partial u}{\partial y}=\dfrac{1}{4uv-1}$.

## THEOREM ON TOTAL DIFFERENTIALS

NOTES

Let $$u = f(x, y) \qquad ...(1)$$
be a function of $x, y$ which possesses continuous partial derivatives of first order w.r.t. $x$ and $y$ in the domain of definition of the function.

Let $x, y$ receive increments $\delta x, \delta y$ and let $\delta u$ be the consequent change in $u$, then we have

$$u + \delta u = f(x + \delta x, y + \delta y) \qquad ...(2)$$

$$\therefore \quad \delta u = f(x + \delta x, y + \delta y) - f(x, y)$$
$$= [f(x + \delta x, y + \delta y) - f(x, y + \delta y)] + [f(x, y + \delta y) - f(x, y)]$$

Applying Lagrange's Mean-value Theorem to each of the two differences, we get

$$\delta u = \delta x \,.\, f_x(x + \theta_1 \delta x, y + \delta y) + \delta y + f_y (x, y + \theta_2 \delta y) \qquad ...(3)$$

where $$0 < \theta_1 < 1,\ 0 < \theta_2 < 1$$

But $f_x(x, y)$ is given to be continuous

$$\therefore \quad \text{Lt}\, f_x(x + \theta_1 \delta x, y + \delta y) = f_x(x, y)$$

$\therefore$ There exists a + ve number $\varepsilon_1$ s.t.

$$f_x(x + \theta_1 \delta x, y + \delta y) = f_x(x, y) + \varepsilon_1.$$

Similarly, there exists a + ve number $\varepsilon_2$ s.t.

$$f_y(x, y + \theta_2 \delta y) = f_y(x, y) + \varepsilon_2$$

$\therefore$ (3) becomes $\delta u = \delta x[f_x(x, y) + \varepsilon_1] + \delta y[f_y(x, y) + \varepsilon_2]$

$$= \left[\frac{\partial u}{\partial x} . \delta x + \frac{\partial u}{\partial y} . \delta y\right] + [\varepsilon_1 \delta x + \varepsilon_2 \delta y] \qquad \Bigg| \because\ f_x = \frac{\partial u}{\partial x} \text{ and } f_y = \frac{\partial u}{\partial y}$$

Thus, the change $\delta u$ in $u$ consists of two parts as shown in brackets, of these the first is called the *Differential* of $u$ and is denoted by $du$. Hence,

$$du = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} . \delta y \qquad ...(4)$$

Let $u = x$, then by (4) $dx = du = 1 \,.\, \delta x + 0.\ \delta y = \delta x$

Similarly, by taking $u = y$, we prove that $\delta y = dy$.

Thus, (4) takes the form $\mathbf{du} = \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}} \mathbf{dx} + \dfrac{\partial \mathbf{u}}{\partial \mathbf{y}} \mathbf{dy}.$

**Note.** The differentials $dx$ and $dy$ of the independent variables $x$ and $y$ are the *actual changes* $\delta x$ and $\delta y$ but the differential $du$ of the dependent variable $u$ is not the same as the change $\delta u$ ; it being the *principal part* of the increment $\delta u$.

## SOLVED EXAMPLES

**Example 17.** *If the sides an angles of a triangle ABC vary in such a way that its circumradius remains constant, prove that*

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0\ .$$

**Sol.** We know that circumradius R of a triangle ABC is given by

$$\frac{a}{\sin \mathbf{A}} = \frac{b}{\sin \mathbf{B}} = \frac{c}{\sin \mathbf{C}} = 2\mathbf{R} \text{ (constant (given))}$$

$$\therefore \quad a = 2R \sin A,\ b = 2R \sin B,\ c = 2R \sin C$$

NOTES

Taking differentials (By Art. 14)

$$da = 2R \cos A\, dA,\ db = 2R \cos B\, dB,\ dc = 2R \cos C\, dc$$

$$\therefore \quad \frac{da}{\cos A} = 2R\, dA \quad \ldots(1)$$

$$\frac{db}{\cos B} = 2R\, dB \quad \ldots(2)$$

$$\frac{dc}{\cos C} = 2R\, dC \quad \ldots(3)$$

Adding equations (1), (2) and (3), we have

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \quad \ldots(4)$$

Because A, B, C are the angles of a triangle

$\therefore$ $A + B + C = \pi$

Taking differentials $dA + dB + dC = 0$

Putting this value of $dA + dB + dC = 0$ in (4)

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

## EXERCISE D

**1.** (*a*) If $z = \log (x^2 + xy + y^2)$, prove that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2$.

(*b*) If $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

**2.** (*a*) If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

(*b*) If $u = x^2(y - z) + y^2(z - x) + z^2(x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

(*c*) If $u = x^2y + y^2z + z^2x$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$.

**3.** (*a*) If $u = (1 - 2xy + y^2)^{-1/2}$, prove that $x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = y^2 u^3$.

(*b*) If $u = \tan^{-1}\left[\frac{xy}{\sqrt{1 + x^2 + y^2}}\right]$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2 + y^2)^{3/2}}$.

**4.** If $u = e^{x - at} \cos (x - at)$, show that $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$.

**5.** If $u = \log (x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x + y + z)^2}$.

$$\left[\textbf{Hint.} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u \right.$$

$$\left. = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right).\right]$$

NOTES

6. The conduction of heat along a bar satisfies the differential equation

$$\frac{\partial u}{\partial t} = \mu \cdot \frac{\partial^2 u}{\partial x^2}.$$

Show that if $u = A\, e^{-gx} \sin(nt - gx)$, where A, $g$, $n$ are positive constants ; then $g = \sqrt{\dfrac{n}{2\mu}}$.

7. If $f(x, y) = \sqrt{x^2 - y^2} \sin^{-1} \dfrac{y}{x}$, prove that $x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y} = f(x, y)$.

8. If $z = e^{ax + by} f(ax - by)$, show by using the formula of composite differentiation that

$$b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz.$$

9. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\dfrac{du}{dt} = \dfrac{3}{\sqrt{1 - t^2}}$.

[**Hint.** $1 - 9t^2 + 24t^4 - 16t^6 = 1 - t^2 - 8t^2 + 8t^4 + 16t^4 - 16t^6$
$= (1 - t^2) - 8t^2 (1 - t^2) + 16t^4 (1 - t^2)$
$= (1 - t^2)(1 - 8t^2 + 16t^4) = (1 - t^2)(1 - 4t^2)^2$.]

10. (*a*) If $u$ and $v$ are functions of $x$ and $y$ defined by

$x = u + e^{-v} \sin u$, $y = v + e^{-v} \cos u$, prove that $\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$.

[**Hint. 1.** Diff. both equations partially w.r.t. $x$ and eliminate $\dfrac{\partial u}{\partial x}$.

**2.** Diff. both equations partially w.r.t. $y$ and then eliminate $\dfrac{\partial v}{\partial y}$.]

(*b*) If $x$ and $y$ are functions of $u$ and $v$ defined by $u = x + e^{-y} \sin x$, $v = y + e^{-y} \cos x$,

prove that $\dfrac{\partial x}{\partial v} = \dfrac{\partial y}{\partial u}$.

[**Hint**. Same as (*a*) part.]

11. If $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = m(m + 1) r^{m-2}$.

12. If $z = \dfrac{xy}{x - y}$, prove that $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2}{x - y}$.

13. Find the second order partial derivatives of $e^{x^y}$.

## Answer

13. $y e^{x^y} x^{y-2} (yx^y + y - 1)$, $e^{x^y} x^{y-1} [1 + (1 + x^y) \log x^y]$, $e^{x^y} x^y (1 + x^y)(\log x)^2$.

# 7. JACOBIANS

NOTES

**STRUCTURE**

Introduction

If $u$, $v$ are Functions of $r$, $s$ where $r$, $s$ are Functions of $x$, $y$; then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

If $J_1$ is the Jacobian of $u$, v w.r.t. $x$ and $y$ and $J_2$ is the Jacobian of $x$, $y$ w.r.t. $u$ and $v$; then $J_1J_2 = 1$ *i.e.*, $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$

Def. Functional Dependence

Theorem on Functional Dependence

## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Def. Functional Dependence
- Theorem on Functional Dependence

## INTRODUCTION

If $u$ and $v$ are functions of two independent variables $x$ and $y$ ; then the determinant $\begin{vmatrix} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} & \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \\ \frac{\partial \mathbf{v}}{\partial \mathbf{x}} & \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \end{vmatrix}$ is called **Jacobian** of $u$, $v$ with respect to $x$, $y$ and is denoted by the symbol $\mathbf{J}\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial\,(u,v)}{\partial\,(x,y)}$ .

Similarly, if $u$, $v$, $w$ be functions of $x$, $y$, $z$ ; then the Jacobian of $u$, $v$, $w$ with respect to $x$, $y$, $z$ is denoted by $\mathbf{J}\left(\frac{u,v,w}{x,y,z}\right)$ or $\frac{\partial\,(u,v,w)}{\partial\,(x,y,z)}$ and is defined as

$$\frac{\partial (\mathbf{u, v, w})}{\partial (\mathbf{x, y, z})} = \begin{vmatrix} \frac{\partial \mathbf{u}}{\partial \mathbf{x}} & \frac{\partial \mathbf{u}}{\partial \mathbf{y}} & \frac{\partial \mathbf{u}}{\partial \mathbf{z}} \\ \frac{\partial \mathbf{v}}{\partial \mathbf{x}} & \frac{\partial \mathbf{v}}{\partial \mathbf{y}} & \frac{\partial \mathbf{v}}{\partial \mathbf{z}} \\ \frac{\partial \mathbf{w}}{\partial \mathbf{x}} & \frac{\partial \mathbf{w}}{\partial \mathbf{y}} & \frac{\partial \mathbf{w}}{\partial \mathbf{z}} \end{vmatrix}.$$

NOTES

**In general.** If $f_1, f_2, \ldots f_n$ be $n$ functions of $n$ variables $x_1, x_2, \ldots x_n$ possessing partial derivatives of the first order at every point of the domain of definition of the functions, then the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is called the **Jacobian** of $f_1, f_2, \ldots f_n$ w.r.t. $x_1, x_2, \ldots x_n$.

The above Jacobian is denoted by

$$\frac{\partial (f_1, f_2, \ldots f_n)}{\partial (x_1, x_2, \ldots x_n)} \quad \text{or} \quad J(f_1, f_2, \ldots f_n).$$

## SOLVED EXAMPLES

**Example 1.** *If $u = x + y$ and $v = (x + y)^2$ ; evaluate $\frac{\partial (u, v)}{\partial (x, y)}$.*

**Sol.** $u = x + y$ and $v = (x + y)^2$ (given)

$$\therefore \quad \frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = 2(x + y), \ \frac{\partial v}{\partial y} = 2(x + y)$$

We know by def. of Jacobian in Art. 1 that

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Putting values of partial derivatives

$$\begin{vmatrix} 1 & 1 \\ 2(x + y) & 2(x + y) \end{vmatrix} = 2(x + y) - 2(x + y) = 0.$$

**Example 2.** *If $u = x \sin y \cos z$, $v = x \sin y \sin z$, $w = x \cos y$; then find $J\left(\frac{u, v, w}{x, y, z}\right)$.*

**Sol.** $u = x \sin y \cos z$ (given)

Diff. partially w.r.t. $x$, $y$ and $z$ respectively,

$$\frac{\partial u}{\partial x} = \sin y \cos z, \ \frac{\partial u}{\partial y} = x \cos y \cos z, \ \frac{\partial u}{\partial z} = -x \sin y \sin z$$

Again, $v = x \sin y \sin z$ (given)

$$\therefore \quad \frac{\partial v}{\partial x} = \sin y \sin z\,, \quad \frac{\partial v}{\partial y} = x \cos y \sin z\,, \quad \frac{\partial v}{\partial z} = x \sin y \cos z$$

Again, $w = x \cos y$ (given)

$$\therefore \quad \frac{\partial w}{\partial x} = \cos y, \quad \frac{\partial w}{\partial y} = -x \sin y, \quad \frac{\partial w}{\partial z} = 0\,.$$



We know by Def. of Jacobian in Art. 1 that

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Putting values of partial derivatives

$$= \begin{vmatrix} \sin y \cos z & x \cos y \cos z & -x \sin y \sin z \\ \sin y \sin z & x \cos y \sin z & x \sin y \cos z \\ \cos y & -x \sin y & 0 \end{vmatrix}$$

Expanding by first row

$$= \sin y \cos z\,(0 + x^2 \sin^2 y \cos z) - x \cos y \cos z$$
$$(0 - x \sin y \cos y \cos z) - x \sin y \sin z\,(-x \sin^2 y \sin z - x \cos^2 y \sin z)$$
$$= x^2 \sin^3 y \cos^2 z + x^2 \sin y \cos^2 y \cos^2 z + x^2 \sin y \sin^2 z\,(\sin^2 y + \cos^2 y)$$
$$= x^2 \sin y \cos^2 z\,(\sin^2 y + \cos^2 y) + x^2 \sin y \sin^2 z$$
$$= x^2 \sin y\,[\cos^2 z + \sin^2 z] = x^2 \sin y.$$

**Example 3.** *If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$ and $z = r \cos\theta$; evaluate* $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

**Sol.** $x = r \sin\theta \cos\phi$ (given)

$$\therefore \quad \frac{\partial x}{\partial r} = \sin\theta \cos\phi\,, \quad \frac{\partial x}{\partial \theta} = r \cos\theta \cos\phi\,, \quad \frac{\partial x}{\partial \phi} = -r \sin\theta \sin\phi\,.$$

Again $y = r \sin\theta \sin\phi$ (given)

$$\therefore \quad \frac{\partial y}{\partial r} = \sin\theta \sin\phi\,, \quad \frac{\partial y}{\partial \theta} = r \cos\theta \sin\phi\,, \quad \frac{\partial y}{\partial \phi} = r \sin\theta \cos\phi$$

Again $z = r \cos\theta$ (given)

$$\therefore \quad \frac{\partial z}{\partial r} = \cos\theta\,, \quad \frac{\partial z}{\partial \theta} = -r \sin\theta \text{ and } \frac{\partial z}{\partial \phi} = 0$$

We know by def. of Jacobian in Art. 1 that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

NOTES

Putting values of partial derivatives

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Expanding by first row

$$= \sin\theta\cos\phi\,(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi\,(0 - r\sin\theta\cos\theta\cos\phi)$$
$$- r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$
$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin\theta\sin^2\phi\,(\sin^2\theta + \cos^2\theta)$$
$$= r^2\sin\theta\cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi$$
$$= r^2\sin\theta\,(\cos^2\phi + \sin^2\phi) = r^2\sin\theta.$$

**Example 4.** *If* $u_1 = 1 - x_1,\ u_2 = x_1(1 - x_2),\ u_3 = x_1x_2(1 - x_3), \ldots\ldots$
$u_n = x_1x_2 \ldots\ldots x_{n-1}(1 - x_n)$ *; then*

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = (-1)^n\, x_1^{n-1}\, x_2^{n-2} \ldots\ldots x_{n-1}.$$

**Sol.** $u_1 = 1 - x_1$

$$\therefore \quad \frac{\partial u_1}{\partial x_1} = -1,\ \frac{\partial u_1}{\partial x_2} = 0, \ldots \frac{\partial u_1}{\partial x_n} = 0$$

$u_2 = x_1(1 - x_2)$

$$\therefore \quad \frac{\partial u_2}{\partial x_1} = 1 - x_2,\ \frac{\partial u_2}{\partial x_2} = -x_1,\ \frac{\partial u_2}{\partial x_3} = 0, \ldots \frac{\partial u_2}{\partial x_n} = 0$$

$u_3 = x_1x_2(1 - x_3)$

$$\therefore \quad \frac{\partial u_3}{\partial x_1} = x_2(1 - x_3),\ \frac{\partial u_3}{\partial x_2} = x_1(1 - x_3),\ \frac{\partial u_3}{\partial x_3} = -x_1x_2, \ldots \frac{\partial u_3}{\partial x_n} = 0$$

$u_n = x_1x_2 \ldots x_{n-1}(1 - x_n)$

$$\therefore \quad \frac{\partial u_n}{\partial x_1} = x_2 \ldots x_{n-1}(1 - x_n),\ \frac{\partial u_n}{\partial x_2} = x_1x_3 \ldots x_{n-1}(1 - x_n), \ldots$$

$$\frac{\partial u_n}{\partial x_n} = -x_1x_2 \ldots x_{n-1}$$

We know that

$$\frac{\partial(u_1, u_2, \ldots u_n)}{\partial(x_1, x_2, \ldots x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \cdots & \frac{\partial u_3}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Putting values

$$= \begin{vmatrix} -1 & 0 & 0 \ldots\ldots 0 \\ 1 - x_2 & -x_1 & 0 \ldots\ldots 0 \\ x_2(1 - x_3) & x_1(1 - x_3) & -x_1x_2 \ldots\ldots 0 \\ \cdots & \cdots & \cdots \\ x_2 \ldots x_{n-1}(1 - x_n) & x_1x_3 \ldots x_{n-1}(1 - x_n) \quad \ldots\ldots & -x_1x_2 \ldots x_{n-1} \end{vmatrix}$$

= Product of diagonal entries

[∵ determinant of a lower triangular matrix ($a_{ij} = 0$ for $i < j$) is the product of its diagonal elements]

$= (-1)(-x_1)(-x_1x_2)(-x_1x_2x_3) \ldots (-x_1x_2 \ldots x_{n-1})$

$= (-1)^n \, x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$.

## EXERCISE A

**1.** (*a*) If $u = x^2$, $v = y$ ; then prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = 2x$.

(*b*) If $u = x$, $v = y^2$, then prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = 2y$.

(*c*) If $x = u(1 + v)$ and $y = v(1 + u)$, show that $\dfrac{\partial(x, y)}{\partial(u, v)} = 1 + u + v$.

**2.** If $u = x^2 - 2y$, $v = x + y$ ; then prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = 2x + 2$.

**3.** If $u = e^x \sin y$, $v = e^y \cos x$ ; then prove that $\dfrac{\partial(u, v)}{\partial(x, y)} = e^{x+y} \sin(x + y)$ .

**4.** Prove that $J\left(\dfrac{u, v}{x, y}\right)$ at the point (1, 2) $= \dfrac{1}{\sqrt{5}}$

where $u = \sqrt{x^2 + y^2}$ , $v = \tan^{-1} \dfrac{y}{x}$ for $(x, y) \neq (0, 0)$.

[**Hint.** Reproduce Example 3 Page 260. Then put $x = 1$, $y = 2$.]

**5.** (*a*) Calculate the Jacobian $\dfrac{\partial(f, g)}{\partial(x, y)}$ where $f(x, y) = x^2 - x \sin y$ ; $g(x, y) = x^2y^2 + x + y$.

(*b*) If $u = \dfrac{y^2}{2x}$ , $v = \dfrac{x^2 + y^2}{2x}$ ; find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

**6.** If $u = \dfrac{x + y}{1 - xy}$ , $v = \tan^{-1} x + \tan^{-1} y$ ; find $\dfrac{\partial(u, v)}{\partial(x, y)}$.

**7.** If $u = x^2 + y^2 + z^2$, $v = y$, $w = z$ ; then prove that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 2x$.

**8.** If $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ ; then evaluate $\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}$.

**9.** If $u = \dfrac{yz}{x}$ , $v = \dfrac{zx}{y}$ and $w = \dfrac{xy}{z}$ , show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 4$ .

**10.** If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$ ; find $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$.

**11.** If $u = \dfrac{x}{y - z}$ , $v = \dfrac{y}{z - x}$ , $w = \dfrac{z}{x - y}$ ; show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 0$ .

**12.** If $u = xyz$, $v = xy + yz + zx$, $w = x + y + z$ ; show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = (x - y)(y - z)(z - x)$ .

NOTES

**13.** If $y_1 = x_1^3$, $y_2 = e^{x_2}$ and $y_3 = x_1 + \sin x_3$ ; find $\dfrac{\partial (y_1, y_2, y_3)}{\partial (x_1, x_2, x_3)}$.

**14.** If F = $xu + v - y$, G = $u^2 + vy + w$, H = $zu - v + vw$ ; compute $\dfrac{\partial (F, G, H)}{\partial (u, w, v)}$.

[**Hint.** $\dfrac{\partial F}{\partial u} = x, \dfrac{\partial F}{\partial w} = 0, \dfrac{\partial F}{\partial v} = 1, \dfrac{\partial G}{\partial u} = 2u, \dfrac{\partial G}{\partial w} = 1, \dfrac{\partial G}{\partial v} = y, \dfrac{\partial H}{\partial u} = z, \dfrac{\partial G}{\partial w} = v, \dfrac{\partial G}{\partial v} = -1 + w$

$$\therefore \quad \frac{\partial (F, G, H)}{\partial (u, w, v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial w} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial w} & \frac{\partial G}{\partial v} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial w} & \frac{\partial H}{\partial v} \end{vmatrix} .]$$

**15.** Find the Jacobian of $u, v, w$ w.r.t. $x, y, z$ given that $u = x + y + z$, $v^2 = yz + zx + xy$ and $w^3 = xyz$.

[**Hint.** $u = x + y + z \Rightarrow \dfrac{\partial u}{\partial x} = 1$ etc.

$v^2 = yz + zx + xy \Rightarrow 2v \dfrac{\partial v}{\partial x} = z + y \quad \therefore \dfrac{\partial v}{\partial x} = \dfrac{y + z}{2v}$ etc.

$w^3 = xyz \Rightarrow 3w^2 \dfrac{\partial w}{\partial x} = yz \quad \therefore \dfrac{\partial w}{\partial x} = \dfrac{yz}{3w^2}$ etc.]

### Answers

**5.** (*a*) $(2x - \sin y)(2x^2 y + 1) + x \cos y (2x y^2 + 1)$ (*b*) $-\dfrac{y}{2x}$

**6.** 0 **8.** $r$ **10.** $10x + 4$

**13.** $3 x_1^2 e^{x_2} \cos x_3$ **14.** $xw - x - xyv + 2uv - z$

**16.** $\dfrac{-(x - y)(y - z)(z - x)}{6vw^2}$.

## If u, v are functions of r, s where r, s are functions of x, y ; then $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{\partial(u, v)}{\partial(r, s)} \times \dfrac{\partial(r, s)}{\partial(x, y)}$

**Proof.** Because $u, v$ are functions of $r, s$ and $r, s$ are functions of $x, y$ ; therefore $u, v$ are composite functions of $x, y$.

Therefore,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_r r_x + u_s s_x \quad ...(1)$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = u_r r_y + u_s s_y \quad ...(2)$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = v_r r_x + v_s s_x \quad ...(3)$$
$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = v_r r_y + v_s s_y \quad ...(4)$$

Now, R.H.S. $= \dfrac{\partial(u,v)}{\partial(r,s)}\dfrac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial s} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial s} \end{vmatrix}\begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix}\begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

Performing row by column multiplication,

$$= \begin{vmatrix} u_r\, r_x + u_s\, s_x & u_r\, r_y + u_s\, s_y \\ v_r\, r_x + v_s\, s_x & v_r\, r_y + v_s\, s_y \end{vmatrix}$$

Putting values from (1), (2), (3) and (4)

$$= \begin{vmatrix} \dfrac{\partial \mathbf{u}}{\partial \mathbf{x}} & \dfrac{\partial \mathbf{u}}{\partial \mathbf{y}} \\ \dfrac{\partial \mathbf{v}}{\partial \mathbf{x}} & \dfrac{\partial \mathbf{v}}{\partial \mathbf{y}} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \text{L.H.S.}$$

## If $J_1$ is the Jacobian of u, v w.r.t. x and y and $J_2$ is the Jacobian of x, y w.r.t. u and v ; then $J_1J_2 = 1$ i.e., $\dfrac{\partial(u,v)}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(u,v)} = 1$

**Proof.** Because $J_1$ is the Jacobian of $u$, $v$ w.r.t. $x$ and $y$ (given), therefore $u$ and $v$ are functions of $x$ and $y$. So, let $u = u(x, y)$ and $v = v(x, y)$. Again because $J_2$ is the Jacobian of $x$ and $y$ w.r.t. $u$ and $v$ ; therefore $x$ and $y$ are functions of $u$ and $v$.

Combining the two ; $u$ and $v$ are composite functions of $u$ and $v$.

Differentiating $u = u(x, y)$ partially w.r.t. $u$ and $v$ ; we have

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = u_x\, x_u + u_y\, y_u \qquad \text{...(1)}$$

$$0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = u_x\, x_v + u_y\, y_v \qquad \text{...(2)}$$

Again differentiating $v = v(x, y)$ partially w.r.t. $u$ and $v$ ;

$$0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} = v_x\, x_u + v_y\, y_u \qquad \text{...(3)}$$

$$1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = v_x\, x_v + v_y\, y_v \qquad \text{...(4)}$$

$$\text{L.H.S.} = \frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Performing row by column multiplication

$$= \begin{vmatrix} u_x\, x_u + u_y\, y_u & u_x\, x_v + u_y\, y_v \\ v_x\, x_u + v_y\, y_u & v_x\, x_v + v_y\, y_v \end{vmatrix}$$

Putting values from (1), (2), (3) and (4), we have

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 = \text{R.H.S.}$$

**NOTES**

*i.e.,* $$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1 \text{ or } J_1J_2 = 1.$$

**Cor.** $\therefore$ $$\frac{\partial(\mathbf{u},\mathbf{v})}{\partial(\mathbf{x},\mathbf{y})} = \frac{\mathbf{1}}{\left(\frac{\partial(\mathbf{x},\mathbf{y})}{\partial(\mathbf{u},\mathbf{v})}\right)}$$

## SOLVED EXAMPLES

**Example 5.** *If* $u = x(1 - y)$, $v = xy$ ; *prove that* $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$.

**Sol.** $u = x(1 - y) = x - xy$ ...(1) $v = xy$ ...(2) (given)

$$\therefore \quad \frac{\partial u}{\partial x} = 1 - y, \quad \frac{\partial u}{\partial y} = -x,$$

$$\frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x$$

$$\therefore \quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x(1-y) + xy = x - xy + xy = x \quad \text{...(3)}$$

To find $\frac{\partial(x,y)}{\partial(u,v)}$, let us make $x$ and $y$ as functions of $u$, $v$.

Adding eqn. (1) and (2), we have

$$u + v = x$$

Putting $x = u + v$ in (2), $v = (u + v)y$ $\therefore$ $y = \frac{v}{u+v}$

$$\therefore \quad x = u + v \quad \text{...(4)} \qquad y = \frac{v}{u+v} \quad \text{...(5)}$$

$$\therefore \quad \frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = \frac{(u+v)\,0 - v\,.1}{(u+v)^2} = \frac{-v}{(u+v)^2}$$

$$\frac{\partial x}{\partial v} = 1 \qquad \frac{\partial y}{\partial v} = \frac{(u+v) - v}{(u+v)^2} = \frac{u}{(u+v)^2}$$

$$\therefore \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{v}{(u+v)^2} & \frac{u}{(u+v)^2} \end{vmatrix} = \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x} \quad \text{[By (4)]} \quad \text{...(6)}$$

Multiplying eqns. (3) and (6), we have

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = x\,.\frac{1}{x} = 1.$$

## EXERCISE B

NOTES

**1.** If $x = r\cos\theta$, $y = r\sin\theta$; verify that

$$\frac{\partial(x,y)}{\partial(r,\theta)}\,\frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

[**Hint.** Squaring and adding the two equations $r = \sqrt{x^2 + y^2}$. Dividing $\tan\theta = \frac{y}{x}$ ∴ $\theta = \tan^{-1}\frac{y}{x}$ Reproduce Ex. 2 and 3 under Art. 1.]

**2.** If $u = x - y$, $v = x + y$ ; prove that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$

[Use formula of Cor. Art. 3]

**3.** (*a*) If $u = x + y + z$, $uv = y + z$, $uvw = z$; show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

[**Hint.** From given $u - uv = x + y + z - (y + z) = x$, $uv - uvw = y + z - z = y$ and $uvw = z$

∴ $x = u - uv$, $y = uv - uvw$, $z = uvw$.]

(*b*) Find the Jacobian of $u, v, w$ w.r.t. $x, y, z$ when $x + y + z = u$, $y + z = uv$ and $z = uvw$.

**4.** Find the Jacobain of $u, v, w$ w.r.t. $x, y, z$ given that $x = u + v + w$, $y = uv + vw + wu$ and $z = uvw$.

### Answers

**3.** (*b*) $\frac{1}{u^2 v}$ **4.** $\frac{-1}{(u-v)(v-w)(w-u)}$.

## DEF. FUNCTIONAL DEPENDENCE

Let $u_1, u_2, u_3, \ldots u_m$ be $m$ functions of $n$ independent variables $x_1, x_2, \ldots x_n$.

If there exists a relation $F(u_1, u_2, u_3, \ldots\ldots u_m) = 0$ between these $n$ functions ; then the functions $u_1, u_2, \ldots u_m$ are said be **functionally dependent.**

**Note.** If $u_1, u_2, \ldots u_n$ are functionally dependent, then we also say that $u_1, u_2, \ldots\ldots u_n$ are not independent of one another.

For example,

Let $u = x^2$ and $v = x^6$ be two functions of one variable $x$

Now $v = x^6 = (x^2)^3 = u^3$ | ∵ $u = x^2$

or $v - u^3 = 0$ is the functional relation, $F(u, v) = 0$ between the two variables $u$ and $v$.

**Remark.** If $m > n$ *i.e.*, number of functions is greater than the number of variables ; then the functional dependence generally holds.

For example, if $u = f(x, y)$, $v = g(x, y)$ and $w = h(x, y)$ are three functions of two variables (Here $m = 3$, $n = 2$) ;

We can solve any two of the three say $u = f(x, y)$ and $v = g(x, y)$ for $x$ and $y$ and substitute the values of $x$ and $y$ obtained in $w = h(x, y)$ to obtain a relation in $u$, $v$ and $w$.

## THEOREM ON FUNCTIONAL DEPENDENCE

Let $u_1, u_2, u_3, \ldots u_n$ be $n$ functions of $n$ variables $x_1, x_2, x_3, \ldots x_n$.

NOTES

Then the functions $u_1, u_2, \ldots u_n$ are **functionally dependent** *i.e.*, ∃ a functional relation

$$F(u_1, u_2, \ldots u_n) = 0 \text{ iff}$$

$$\frac{\partial (\mathbf{u_1, u_2, \ldots u_n})}{\partial (\mathbf{x_1, x_2, \ldots x_n})} = \mathbf{0} \textbf{ identically.}$$

**Remark.** The proof of the above theorem is beyond the scope of this book and hence is being omitted.

## SOLVED EXAMPLES

**Example 6.** *Show that $u = \sin x + \sin y$, $v = \sin (x + y)$ are not functionally dependent.*

**Sol.** $u = \sin x + \sin y$ $\qquad$ $v = \sin (x + y)$

$$\therefore \quad \frac{\partial u}{\partial x} = \cos x \qquad \frac{\partial v}{\partial x} = \cos (x + y)$$

$$\frac{\partial u}{\partial y} = \cos y \qquad \frac{\partial v}{\partial y} = \cos (x + y)$$

$$\therefore \quad \frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos x & \cos y \\ \cos(x+y) & \cos(x+y) \end{vmatrix}$$

$$= \cos x \cos (x + y) - \cos y \cos (x + y)$$

$$= \cos (x + y) [\cos x - \cos y] \neq 0 \text{ Identically.}$$

∴ By Art. 5, the functions $u$ and $v$ are not functionally dependent.

**Example 7.** *Show that the functions $f_1(x, y, z) = x + 2y + z$, $f_2(x, y, z) = x - 2y + 3z$ and $f_3(x, y, z) = 2xy - xz + 4yz - 2z^2$ are functionally related.*

*Also find the relation between them.*

**Sol.** ∵ $\quad f_1 = x + 2y + z \qquad \ldots(1)$

$$\therefore \quad \frac{\partial f_1}{\partial x} = 1, \ \frac{\partial f_1}{\partial y} = 2, \ \frac{\partial f_1}{\partial z} = 1$$

Again, ∵ $\quad f_2 = x - 2y + 3z \qquad \ldots(2)$

$$\therefore \quad \frac{\partial f_2}{\partial x} = 1, \ \frac{\partial f_2}{\partial y} = -2, \ \frac{\partial f_2}{\partial z} = 3$$

Again, ∵ $\quad f_3 = 2xy - xz + 4yz - 2z^2 \qquad \ldots(3)$

$$\therefore \quad \frac{\partial f_3}{\partial x} = 2y - z, \ \frac{\partial f_3}{\partial y} = 2x + 4z, \ \frac{\partial f_3}{\partial z} = -x + 4y - 4z$$

We know that
$$\frac{\partial (f_1, f_2, f_3)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

Putting values of partial derivatives

$$= \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

Expanding by first row

$$= 1[2x - 8y + 8z - 6x - 12z] - 2[-x + 4y - 4z - 6y + 3z] + 1[2x + 4z + 4y - 2z]$$

$$= -4x - 8y - 4z - 2(-x - 2y - z) + 2x + 4y + 2z$$

$$= -4x - 8y - 4z + 2x + 4y + 2z + 2x + 4y + 2z = 0$$

$\therefore$ Functions $f_1, f_2, f_3$ are functionally dependent.

**To find the relation between $f_1, f_2, f_3$ ; let us eliminate x, y, z from (1), (2) and (3).**

Squaring Eqns. (1) and (2) and subtracting, we have

$$f_1^2 - f_2^2 = (x + 2y + z)^2 - (x - 2y + 3z)^2$$

$$= (x^2 + 4y^2 + z^2 + 4xy + 4yz + 2xz) - (x^2 + 4y^2 + 9z^2 - 4xy - 12yz + 6xz)$$

$$= -8z^2 + 8xy + 16yz - 4xz$$

$$= 4(2xy - xz + 4yz - 2z^2) = 4f_3. \qquad \text{[By (3)]}$$

or $\quad f_1^2 - f_2^2 = 4f_3$

which is the required relation between the given functions $f_1$, $f_2$ and $f_3$.

## EXERCISE C

**1.** Let $f_1(x, y) = \dfrac{x + y}{1 - xy}$ and $f_2(x, y) = \tan^{-1} x + \tan^{-1} y$ be two functions. Are $f_1(x, y)$ and $f_2(x, y)$ functionally related ?

**2.** Show that the functions

$u = x + y - z$, $v = x - y + z$, $w = x^2 + y^2 + z^2 - 2yz$ are not independent of each other. Also find the relation between them.

**3.** Show that the functions

$u = 3x + 2y - z$, $v = x - 2y + z$ and $w = x(x + 2y - z)$ are not independent and find the relation between them.

**4.** If $u = x + y + z$, $v = xy + yz + zx$, $w = x^3 + y^3 + z^3 - 3xyz$ ; Show that $u$, $v$ and $w$ are connected by a functional relation and find it.

[**Hint.** $w = x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - (xy + yz + zx))$

$= (x + y + z)[(x + y + z)^2 - 2(xy + yz + zx) - (xy + yz + zx)]$

$= (x + y + z)[(x + y + z)^2 - 3(xy + yz + zx)] = u(u^2 - 3v)$.]

**5.** If $u = \dfrac{x}{y - z}$, $v = \dfrac{y}{z - x}$, $w = \dfrac{z}{x - y}$ ; prove that the above functions are not independent and find the relation between them.

[**Hint.** Find $uv + vw + wu$.]

**6.** Show that the functions $u = x^2 + y^2 + z^2$, $v = xy - xz - yz$, $w = x + y - z$ are dependent.

### Answers

**1.** Yes ; $f_1$ and $f_2$ are functionally dependent.

**2.** $u^2 + v^2 = 2w$ **3.** $u^2 - v^2 = 8w$

**4.** $w = u^3 - 3uv$ **5.** $uv + vw + wu + 1 = 0$.

# UNIT 4

NOTES

# 8. TANGENTS AND NORMALS

**STRUCTURE**

Geometrical Interpretatino of $\frac{dy}{dx}$

Find the Equation of the Tangent at any Point of the Curve $y = f(x)$

Find the Equation of the Normal at any Point of the Curve $y = f(x)$

Parametric Coordinates

Angle of Intersection of Two Curves

Polar Coordinates

For any Point $(r, \theta)$ of the Curve $r = f(\theta)$, the angle $\phi$ between the Radius Vector and the Tangent is given by $\tan \phi = f \frac{d\theta}{dr}$

## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Parametric Coordinates
- Angle of Intersection of Two Curves
- Polar Coordinates

## GEOMETRICAL INTERPRETATION OF $\frac{dy}{dx}$

Let $y = f(x)$ be a continuous function of $x$.

Let the curve AB represent graphically the function $y = f(x)$.

Let P$(x, y)$ be any point on the curve.

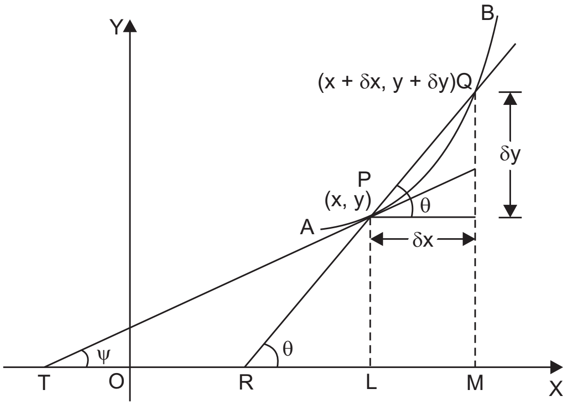
Let Q$(x + \delta x, y + \delta y)$ be a point in the neighbourhood of P.

Join QP and produce it to meet the axis of $x$ in R

Let $\angle XRP = \theta$. Draw PL and QM $\perp$ OX and PN $\perp$ MQ

Now $\quad PN = LM = OM - OL = x + (\delta x) - x = \delta x$

$NQ = MQ - MN = MQ - LP = y + (\delta y) - y = \delta y$




$\angle NPQ = \angle XRP = \theta$

$$\therefore \quad \tan\theta = \frac{NQ}{PN} = \frac{\delta y}{\delta x} \qquad \ldots(i)$$

Now as $Q \to P$, $\delta x \to 0$, secant PQ becomes the tangent PT at P and $\theta \to \psi$ where $\psi = \angle XTP$

$$\therefore \quad \tan\psi = \underset{Q \to P}{\text{Lt}} \tan\theta = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} \qquad \text{[From } (i)\text{]}$$

$$= \frac{dy}{dx}$$

Hence $\frac{dy}{dx} = \tan\psi$ = slope of the tanget at P($x, y$).

Remember 1. The angle which a tangent makes with positive direction of $x$-axis is denoted by $\psi$

2. $\tan\psi$ = slope of the tangent

$$\therefore \quad \frac{dy}{dx} = \tan\psi = \text{slope of the tangent at } (x, y)$$

**Cor. 1.** *If the tangent is parallel to x-axis then $\psi = 0$*

$$\therefore \quad \frac{dy}{dx} = \tan\psi = \tan 0 = 0$$

**Cor. 2.** *If the tangent is perpendicular to x-axis then $\psi = 90°$*

$$\therefore \quad \frac{dy}{dx} = \tan\psi = \tan 90° = \infty \text{ or } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = 0$$

## FIND THE EQUATION OF THE TANGENT AT ANY POINT OF THE CURVE y = f(x)

Let P($x_1$, $y_1$) be any point on the curve $y = f(x)$

Slope of the tangent at P($x_1$, $y_1$) is the value of $\frac{dy}{dx}$ at this point.

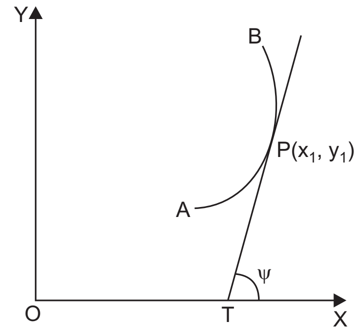
Let this value be denoted by $m$.

NOTES

Now the tangent at P is a line through P($x_1$, $y_1$) having slope $m$.

∴ Equation of the tangent at P is

$y - y_1 = m(x - x_1)$ | Using point-slope from




Note. Equation of tangent to the curve $y = f(x)$ at the point $(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

Since $(x, y)$ is a point on the curve, we use X and Y for the current coordinates.

**Rule to find the equation of the tangent at a point**

1. Find $\frac{dy}{dx}$ from the equation of the curve. This gives the slope of the tangent at the general point $(x, y)$.
2. Find the value of $\frac{dy}{dx}$ at the given point $(x_1, y_1)$. This gives the slope of the tangent at $(x_1, y_1)$
3. Now equation of the tangent is $y - y_1 = m(x - x_1)$ where $m$ denotes the slope found in step 2.

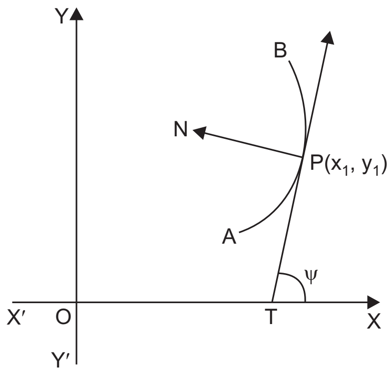
## FIND THE EQUATION OF THE NORMAL AT ANY POINT OF THE CURVE y = f(x)

The normal to a curve atany point P($x_1$, $y_1$) is the straight line through the point perpendicular to the tangent to the curve at that point

Let the slope of the tangent to the curce at P($x_1$, $y_1$) be $m$.

Then slope of the normal to the curve at

$$P(x_1, y_1) = -\frac{1}{m} \qquad | \text{ –ve normal}$$




∴ Equation of the normal to the curve at P($x_1$, $y_1$) is

$$y - y_1 = \frac{1}{m}(x - x_1) \qquad \text{(Point-slope form)}$$

or $$(x - x_1) + m(y - y_1) = 0$$

**Rule to find the equation of the normal at a point**

1. Find $\frac{dy}{dx}$ from the equation of the curve This gives the slope of the tangent at te general point $(x, y)$
2. Find the value of $\frac{dy}{dx}$ at the given point $(x_1, y_1)$. This gives the slope of the tangent at the given point.

3. Find the negative recirprocal of the slope of the tangent at $(x_1, y_1)$. This gives the slope of the normal at $(x_1, y_1)$.
4. Now equation of the normal at $(x_1, y_1)$ is $y - y_1 = m(x - x_1)$ where $m$ denotes the slope found in step 3.

NOTES

## SOLVED EXAMPLES

**Example 1.** *In the curve $3b^2y = x^3 - 3ax^2$, find the points atwhich the tangent is prallel to the axis of x.*

**Sol.** Equation of the curve is

$$3b^2y = x^3 - 3ax^2 \qquad \ldots(1)$$

Differentiating w.r.t. $x$,

$$3b^2\frac{dy}{dx} = 3x^2 - 6ax$$

$$\therefore \qquad \frac{dy}{dx} = \frac{x^2 - 2ax}{b^2} = \frac{x(x-2a)}{b^2}$$

$$= \text{Slope of tangent at } (x, y)$$

If the tangent is parallel to the axis of $x$,

$$\frac{dy}{dx} = 0 \qquad \therefore \quad x = 0, 2a$$

When $x = 0$, from (1), $y = 0$

When $x = 2a$, from (1),

$$3b^2y = 8a^3 - 12a^3 = -4a^3$$

$$\therefore \qquad y = \frac{4a^3}{3b^2}$$

Hence the required points are (0, 0) and $\left(2a, -\frac{4a^3}{3b^2}\right)$

**Example 2.** *Find the point on the curve*

$$x = a(\theta + sin\ \theta),\ y = a(1 - cos\ \theta)$$

*where the tangent is perpendicualr to x-axis.* $(0 \le \theta \le 2\pi)$

**Sol.** The equations of the curve are

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

Differentiating w.r.t. θ, we have

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

If the tangent is perpendicular to the $x$-axis, then $\psi = \frac{\pi}{2}$ so that

$$\frac{dy}{dx} = \tan y = \infty$$

$$\Rightarrow \qquad \tan\frac{\theta}{2} = \infty \quad \Rightarrow \quad \frac{\theta}{2} = \frac{\pi}{2} \quad \therefore \quad \theta = \pi$$

NOTES

$$x = a(\pi + \sin \pi) = a\pi$$
$$y = a(1 - \cos \pi) = a[1 - (-1)] = 2a$$

Hence the required point is ($a\pi$, $2a$).

**Example 3.** *Find the point on the curve y = 3x2 – 2x – 4 at which tangent is perpendicular to the line x + 10y – 7 = 0.*

**Sol.** Equation of the curve is

$$y = 3x^2 - 2x - 4 \qquad \ldots(1)$$

$$\therefore \qquad \frac{dy}{dx} = 6x - 2$$

This is the slope of the tangent at ($x$, $y$)

Slope of the line $x + 10y - 7 = 0$ is $-\frac{1}{10}$ $\left[m = -\frac{\text{co-eff of } x}{\text{co-eff of } y}\right]$

Since the tangent is perpendicular to the given line

$$\therefore \qquad (6x-2)\left(-\frac{1}{10}\right) = -1 \qquad [m_1 m_2 = -1]$$

$$\Rightarrow \qquad 6x - 2 = 10 \quad \therefore \quad x = 2$$

Putting $x = 2$ in (1), we have

$$y = 12 - 4 - 4 = 4$$

$\therefore$ The required point is (2, 4)

**Example 4.** *Find the equation of the normal at (a, a) to the curve $x^2y^3 = a^5$.*

**Sol.** Equation of the curve is $x^2y^3 = a^5$

Differentiating both sides w.r.t. $x$,

$$2xy^3 + 3x^2y^2\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{2y}{3x}$$

$$\text{Value of } \frac{dy}{dx} \text{ at } (a, a) = -\frac{2a}{3a} = -\frac{2}{3}$$

$$\therefore \quad \text{Slope of tangent at } (a, a) = -\frac{2}{3}$$

$$\text{Slope of normal at } (a, a) = \frac{3}{2}$$

Equation of normal at ($a$, $a$) is

$$y - a = \frac{3}{2}(x - a) \text{ or } 3x - 2y - a = 0$$

**Example 5.** *The equation to the tangent at the point (2, 3) on the curve $y^2 = ax^3 + b$ is y = 4x – 5. Find the values of a and b.*

**Sol.** Please try yourself Eq. of tangent at (2, 3) is $y = 2ax - 4a + 3$

$$\therefore \qquad 2a = 4 \text{ and } -4a + 3 = -5 \quad \Rightarrow \quad a = 2$$

$\because$ Equation of curve is $y^2 = 2x^3 + b$

$\because$ The point (1, 1) lies on it $\therefore$ $1 = 2 + b \Rightarrow b = -1$

**Example 6.** *Find the points on the curve*

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

*where the tangent is parallel to y = 2x. Also prove that the two these points have the same tangent.*

**Sol.** Equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$\because$ The tangent is parallel to the line $y = 2x$

$\therefore$ Slope of tangent = slope of line = 2

*i.e.*, $\frac{dy}{dx} = 2 \quad \therefore \quad 4x^3 - 18x^2 + 26x - 10 = 2$

or $\quad 2x^3 - 9x^2 + 13x - 6 = 0$ ...(*ii*)

By inspection of $x = 1$ satisfies it

$\therefore$ Dividing L.H.S of (*ii*) by $(x - 1)$ by synthetic division

$$\begin{array}{c|cccc} 1 & 2 & -9 & 13 & -6 \\ & & 2 & -7 & 6 \\ \hline & 2 & -7 & 6 & \underline{|0} \end{array}$$

Depressed equation is $2x^2 - 7x + 6 = 0$

$$x = \frac{7 \pm \sqrt{49-48}}{4} = \frac{7 \pm 1}{4} = 2, \frac{3}{2}$$

$\therefore$ Abscissae of points, tangents at which are parallel to given line, are $1, 2, \frac{3}{2}$.

when $x = 1$, from (*i*), $\quad y = 1 - 6 + 13 - 10 + 5 = 3$

when $x = 2$, from (*i*), $\quad y = 16 - 48 + 52 - 20 + 5 = 5$

when $x = \frac{3}{2}$, from (*i*), $\quad y = \frac{81}{16} - \frac{81}{4} + \frac{117}{4} - 15 + 5 = \frac{65}{16}$

The requried points are A(1, 3); B(2, 5); $C\left(\frac{3}{2}, \frac{65}{16}\right)$

(*b*) $\because$ Tangents at A, B, C are parallel to the given line

$\therefore$ Slope of tangent at A, B, C = 2

Equation of tangent at A is $y - 3 = 2(x - 1)$

*i.e.*, $\quad 2x - y + 1 = 0$

Equation of tangent at B is $y - 5 = 2(x - 2)$

*i.e.*, $\quad 2x - y + 1 = 0$

$\therefore$ The points A and B have the same tangent $2x - y + 1 = 0$.

**Example 7.** *For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.*

**Sol.** Equation of curve is

$$y = 4x^3 - 2x^5 \quad \text{...(i)}$$

Let the tangent at $(x_1, y_1)$ pass through the origin

$$\frac{dy}{dx} = 12x^2 - 10x^4$$

Slope of tangent at $(x_1, y_1) = 12x_1^2 - 10x_1^4$

Equation of the tangent at $(x_1, y_1)$ is

$$y - y_1 = (12x_1^2 - 10x_1^4)(0 - x_1)$$

$$y_1 = 12x_1^3 - 10x_1^5 \quad \ldots(ii)$$

Also $(x_1, y_1)$ lies on the curve $(i)$

$$\therefore \quad y_1 = 4x_1^3 - 2x_1^5 \quad \ldots(iii)$$

Subtracting (3) from (2),

$$0 = 8x_1^3 - 8x_1^5$$

or $\quad 8x_1^3(1 - x_1^2) = 0 \quad \therefore \quad x_1 = 0, \pm 1$

when $\quad x_1 = 0$, from (3), $\quad y_1 = 0$

when $\quad x_1 = 1$, from (3), $\quad y_1 = 2$

when $\quad x_1 = -1$, from (3), $\quad y_1 = -2$

Hence the required points are (0, 0), (1, 2) and (– 1, – 2).

**Example 8.** *Find the equation of the tangent line to the curve* $y = \sqrt{5x-3} - 2$ *which is*

(*i*) *parallel to the line* $2x - y + 9 = 0$

(*ii*) *perpendicular to the line* $5y + 2\sqrt{2x} = 13$

**Sol.** Equation of the curve is $y = \sqrt{5x-3} - 2 \quad \ldots(1)$

$$\frac{dy}{dx} = \frac{1}{2}(5x-3)^{-1/2} \cdot \frac{d}{dx}(5x-3) = \frac{5}{2\sqrt{5x-3}}$$

(*i*) Given line is $2x - y + 9 = 0 \quad \ldots(2)$

Its slope = 2

If the tangent is parallel to (2), then slope of tangent is 2

$$\therefore \quad \frac{5}{2\sqrt{5x-3}} = 2, \quad \text{Squaring } 25 = 16(5x-3)$$

or $\quad x = \dfrac{73}{80}$

Putting $x = \dfrac{73}{80}$ in (1), $\quad y = \sqrt{\dfrac{73}{16} - 3} - 2 = \dfrac{5}{4} - 2 = \dfrac{3}{4}$

Thus at the point $\left(\dfrac{73}{80}, -\dfrac{3}{4}\right)$ the tangent is parallel to (2)

Equation of tangent is $y + \dfrac{3}{4} = 2\left(x - \dfrac{73}{80}\right)$ or $80x - 40y = 103$

(*ii*) Given line is $5y + 2\sqrt{2}x = 13 \quad \ldots(3)$

Its slope $= -\dfrac{2\sqrt{2}}{5}$

If the tangent is perpendicular to (3), then slope of tangent is $\dfrac{5}{2\sqrt{2}}$, negative reciprocal of slope of (3).

$\therefore \qquad \dfrac{5}{2\sqrt{5x-3}} = \dfrac{5}{2\sqrt{2}}$ or $\sqrt{5x-3} = \sqrt{2}$

or $\qquad 5x - 3 = 2 \quad \therefore \quad x = 1$

Putting $x = 1$ in (1), $y = \sqrt{2} - 2$

Thus at the point $(1, \sqrt{2} - 2)$, the tangent is perpendicular to (3)

Equation of tangent is

$$y - (\sqrt{2} - 2) = \frac{5}{2\sqrt{2}}(x - 1)$$

or $\qquad 2\sqrt{2}y - 4 + 4 + 4\sqrt{2} = 5x - 5$

or $\qquad 2\sqrt{2}y - 5x + 4\sqrt{2} + 1 = 0.$

**Example 9.** *Prove that the equation of the tangent at the point $(4m^2, 8m^3)$ of the curve $y^2 = x^3$ is $y = 3mx - 4x^3$ and that it meets the curve again in the point $(m^2, -m^3)$. Show that if $9m^2 = 2$, tangent is also a normal to the curve.*

**Sol.** Equation of the curve is $y^2 = x^2$

Differentiating, $\qquad 2y\dfrac{dy}{dx} = 3x^2 \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{3x^2}{2y}$

Value of $\dfrac{dy}{dx}$ at $(4m^2, 8m^3) = \dfrac{3.16m^4}{2.8^3} = 3m$

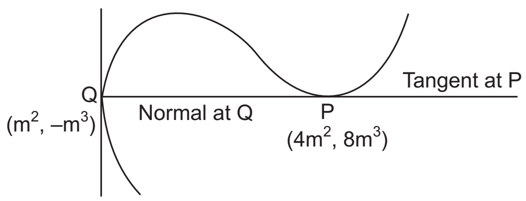
$\therefore$ Slope of tangent at $(4m^2, 8m^3)$ is $3m$

Equation of tangent at $(4m^2, 8m^3)$ is $y - 8m^3 = 3m(x - 4m^2)$

or $\qquad y = 3mx - 4m^3 \qquad \ldots(ii)$

(*ii*) meets (*i*) where (eliminating $y$)

$$(3mx - 4m^3)^2 = x^3$$




or $\qquad x^3 - 9m^2x^2 + 24m^4x - 16m^6 = 0$

or $\qquad (x^3 - m^2)(x^2 - 8m^2x + 16m^4) = 0$ or $(x - m^2)(x - 4m^2)^2 = 0$

or $\qquad x = m^2, 4m^2, 4m^2$

$\therefore$ From (*ii*), $\qquad y = -m^3, 8m^3, 8m^3.$

(*ii*) meets (*i*) in two co-incident points $(4m^2, 8m^3)$ and its therefore, a tangent at $(4m^2, 8m^3)$

The third point of intersection is $(m^2, -m^3)$

Slope of tangent at Q$(m^2, -m^3) = \dfrac{3m^4}{-2m^3} = \dfrac{-3m}{2}$

$\therefore$ Slope of normal at Q $= \dfrac{2}{3m}$

NOTES

If tangent at P is normal to the curve at Q, then

$$3m = \frac{2}{3m} \quad \therefore \quad 9m^2 = 2$$

**Example 10.** *Show that the length of the portion of the tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between the co-ordinate axes is constant.*

**Sol.** Equation of curve is $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating, $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$

or $$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

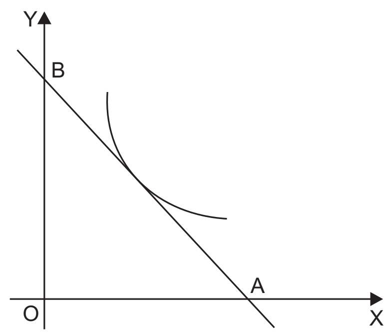
Slope of tangent at $(x, y) = -\frac{y^{1/3}}{x^{1/3}}$

Equation of tangent at $(x, y)$ is

$$Y - y = -\frac{y^{1/3}}{x^{1/3}}(X - x)$$

or $$\frac{Y}{y^{1/3}} - \frac{y}{y^{1/3}} = \frac{X}{x^{1/3}} + \frac{x}{x^{1/3}}$$

or $$\frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} = x^{2/3} + y^{2/3} = a^{2/3} \qquad | \because \text{ of } (i)$$




$$OA = \text{intercept on } x\text{-axis} = a^{2/3}\, x^{1/3}$$

$$OB = \text{intercept on } y\text{-axis} = a^{2/3}y^{1/3}$$

$$\text{Required length} = AB = \sqrt{OA^2 + OB^2} = \sqrt{a^{4/3}(x^{2/3} + y^{2/3})}$$

$$= \sqrt{a^{4/3} \cdot a^{2/3}}$$

$$= \sqrt{a^2} = a \text{ which is constant}$$

**Example 11.** *Prove that all points of the curve*

$$y^2 = 4a\left[x + a \sin \frac{x}{a}\right]$$

*at which the tangent is parallel to the axis of x lie on a parabola.*

**NOTES**

**Sol.** Let $(x_1, y_1)$ be a point on $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ ...(*i*)

the tangent at which is parallel to $x$-axis

$$\therefore \quad y_1^2 = 4a\left(x_1 + a\sin\frac{x_1}{a}\right) \quad ...(ii)$$

Diff. (*i*) $\quad 2y\cdot\frac{dy}{dx} = 4a\left(1 + a\cos\frac{x}{a}\cdot\frac{1}{a}\right)$

or $\quad \frac{dy}{dx} = \frac{2a}{y}\left(1 + \cos\frac{x}{a}\right)$

$\therefore$ Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1}\left(1 + \cos\frac{x_1}{a}\right)$

Since tangent at $(x_1, y_1)$ is parallel to $x$-axis

$\therefore$ Slope of tangent at $(x_1, y_1) = 0$

$$\therefore \quad \frac{2a}{y_1}\left(1 + \cos\frac{x_1}{a}\right) = 0 \quad \text{or} \quad 1 + \cos\frac{x_1}{a} = 0$$

$$\cos\frac{x_1}{a} = -1$$

$$\therefore \quad \sin\frac{x_1}{a} = \sqrt{1 - \cos^2\frac{x_1}{a}} = \sqrt{1-1} = 0$$

$\therefore$ From (*ii*), $\quad y_1^2 = 4ax_1$

$\therefore$ $(x_1, y_1)$ lies on $y^2 = 4ax$, which is a parabola.

**Example 12.** *Prove that the line* $lx + my + n = 0$ *is a normal to* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *if*

$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}.$$

**Sol.** Let the line $lx + my + n = 0$ ...(1)

be normal to $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(2)

at $(x_1, y_1)$

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \qquad | \because (x_1, y_1) \text{ lies on (2)}]$$

Differentiating (2),

$$\frac{2a}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$\therefore$ Slope of tangent at $(x_1, y_1) = -\frac{b^2x_1}{a^2y_1}$

$\Rightarrow$ Slope of normal at $(x_1, y_1) = \frac{a^2y_1}{b^2x_1}$

Equation of normal to (2) at $(x_1, y_1)$ is

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

NOTES

or
$$\frac{b^2 y}{y_1} - b^2 = \frac{a^2 x}{x_1} - a^2$$

or
$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} - (a^2 - b^2) = 0 \quad \ldots(4)$$

Equations (1) and (4) both represent the equation of normal to (2) at $(x_1, y_1)$

$\therefore$ They are identical. Comparing co-efficients, we have

$$\frac{\frac{a^2}{x_1}}{l} = \frac{-\frac{b^1}{y_1}}{m} = \frac{-(a^2 - b^2)}{n}$$

$$\Rightarrow \qquad x_1 = -\frac{na^2}{l(a^2 - b^2)}, \; y_1 = \frac{nb^2}{m(a^2 - b^2)}$$

Putting the values of $x_1$ and $y_1$ in (3), we have

$$\frac{1}{a^2} \cdot \frac{n^2 a^4}{l^2 (a^2 - b^2)^2} + \frac{1}{b^2} \cdot \frac{n^2 b^4}{m^2 (a^2 - b^2)^2} = 1$$

or
$$\frac{n^2 a^4}{l^2 (a^2 - b^2)^2} + \frac{n^2 b^2}{m^2 (a^2 - b^2)^2} = 1$$

or
$$\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

which is the required condition.

## EXERCISE A

**1.** At what points on the curve $y = \sin x$ is the tangent parallel to the $x$-axis?

**2.** Find the points where the tangent is parallel to $x$-axis and where it is parallel to $y$-axis, for the curve.

**3.** Find the point on the curve $2y = 3 - x^2$ the tangent at which is parallel to the line $x + y = 0$.

**4.** Find the equations of the tangent and the normal to the curve $y = x^3$ at the point (2, 8).

**5.** Find the equation of the tangent at to the curve $2x^2 + 3xy + 5y^2 = 10$ at the point (1, 1).

**6.** Find the equation to the normal at the point $(at^2, 2at)$ of the curve $y^2 = 4ax$.

**7.** Find the co-ordinates of the point on the curve $y = x^2 + 3x + 4$, the tangent at which passes throug the origin.

**8.** Find the equation of the normal to the curve $3x^2 - y^2 = 8$ parallel to the line $x + 3y = 4$.

**9.** Find the equation of the tangent line to the curve $y = x^2 + 4x - 16$ which is parallel to the line $3x - y + 1 = 0$.

**10.** Find the equation of the tangent to $x^3 = ay^2$ at $(4am^2, 8am^3)$ and also the points in which the tangent cuts the curve again.

**11.** Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of $y$.

**12.** Prove that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point $(a, b)$, whatever be the value of $n$.

**13.** Prove that the sum of the intercepts on the co-ordinate axes of any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

**14.** Prove that in the catenary $y = \frac{c}{2}\left[e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right]$ the perpendicular drawn from foot of the ordinate of any point on the curve upon the tangent at the same point is of constant length.

**15.** If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle $\phi$ with the axis of $x$, show that its equation is $y \cos \phi - x \sin \phi = a \cos 2\phi$

**16.** Find the length of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepted between the axes.

**17.** Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on the curve $x^2y^2 = x^2 - y^2$.

**18.** If $x \cos \alpha + y \sin \alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then show that

$$a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$$

**19.** Find the condition that the straight line $x \cos \alpha + y \sin \alpha = p$ may be a tangent to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

**20.** Prove that the condition that $x \cos \alpha + y \sin \alpha = p$ should touch $x^m y^n = a^{m+n}$ is

$$p^{m+n} \cdot m^m \cdot n^n = (m + n)^{m+n} \cdot a^{m+n} \cdot \cos^m \alpha \sin^n \alpha$$

**21.** If $x \cos \alpha + y \sin \alpha = p$ touches the curve $\left(\frac{x}{a}\right)^{n/n-1} + \left(\frac{y}{b}\right)^{n/n-1} = 1$, prove that

$$(a \cos \alpha)^n + (b \sin \alpha)^n = p^n$$

**22.** If $x \cos \alpha + y \sin \alpha = p$ touches the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, show that

$$(al)^{n/n-1} + (bm)^{n/n-1} = 1$$

**23.** If $lx + my = 1$ is a normal to the parabloa $y^2 = 4ax$, prove that $al^3 + 2alm^2 = m^2$.

## PARAMETRIC COORDINATES

### To find the equation of the tangent and nor-mal at any point 't' of the curve given by x = f(t) and y = ϕ(t)

Equations of curve are $\left.\begin{array}{l} x = f(t) \\ t = \phi(t) \end{array}\right]$

Slope of tangent at '$t$' $= \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}$

The point '$t$' means the point $[f(t), \phi(t)]$

Equation of tangent at '$t$' is

$$y - \phi(t) = \frac{\phi'(t)}{f'(t)}[x - f(t)]$$

NOTES

Slope of normal at '$t$' $= -\frac{f'(t)}{\phi'(t)}$

$\therefore$ Equation of normal at '$t$' is

$$y - \phi(t) = -\frac{f'(t)}{\phi'(t)}[x - f(t)]$$

**Example 13.** *Find the equations of tangent and normal at any point of the curve:*

($a$) $x = at^2$, $y = 2at$ ($b$) $x = a(t + \sin t)$, $y = a(1 - \cos t)$

**Sol.** ($a$) Equations of the curve are $\left.\begin{matrix} x = at^2 \\ y = 2at \end{matrix}\right]$

$$\frac{dx}{dt} = 2at; \quad \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$$

Slope of tangent at '$t$' $= \frac{1}{t}$

Equation of tangent at '$t$' *i.e.*, at $(at^2, 2at)$ is

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2$$

or $$ty = x + at^2 \qquad \ldots(i)$$

Slope of normal at '$t$' $= -t$

Equation of normal at '$t$' is

$$y - 2at = -t(x - at^2)$$

or $$tx + y = 2at + at^3 \qquad \ldots(ii)$$

($b$) Equations of curve are $\left.\begin{matrix} x = a(t + \sin t) \\ y = a(1 - \cos t) \end{matrix}\right]$

$$\frac{dx}{dt} = a(1 + \cos t) = 2a\cos^2\frac{t}{2}$$

$$\frac{dy}{dt} = a(\sin t) = 2a\sin\frac{t}{2}\cos^2\frac{t}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \tan\frac{t}{2}$$

Slope of tangent at '$t$' is $\tan\frac{t}{2}$

$\therefore$ Equation of tangent at '$t$' is

$$y - a(1 - \cos t) = \tan\frac{t}{2}[x - a(t + \sin t)]$$

or $$y - 2a\sin^2\frac{t}{2} = \frac{\sin t/2}{\cos t/2}\left[x - at - 2a\sin\frac{t}{2}\cos\frac{t}{2}\right]$$

NOTES

or $$y\cos\frac{t}{2} - 2a\sin^2\frac{t}{2}\cos\frac{t}{2} = x\sin\frac{t}{2} - at\sin\frac{t}{2} - 2a\sin^2\frac{t}{2}\cos\frac{t}{2}$$

or $$x\sin\frac{t}{2} - y\cos\frac{t}{2} = at\sin\frac{t}{2}$$

Slope of normal at '*t*' $= -\dfrac{2}{\tan t/2} = -\cot\dfrac{t}{2}$

Equation of normal at '*t*' is

$$y - a(1-\cos t) = -\cot\frac{t}{2}[x - a(t+\sin t)]$$

or $$y - 2a\sin^2\frac{t}{2} = \frac{\cos t/2}{\sin t/2}\left[x - at - 2a\sin\frac{t}{2}\cos\frac{t}{2}\right]$$

or $$y\sin\frac{t}{2} - 2a\sin^3\frac{t}{2} = -x\cos\frac{t}{2} + at\cos\frac{t}{2} + 2a\sin\frac{t}{2}\cos^2\frac{t}{2}$$

or $$x\cos\frac{t}{2} + y\sin\frac{t}{2} = at\cos\frac{t}{2} + 2a\sin\frac{t}{2}\left(\cos^2\frac{t}{2} + \sin^2\frac{t}{2}\right)$$

or $$x\cos\frac{t}{2} + y\sin\frac{t}{2} = at\cos\frac{t}{2} + 2a\sin\frac{t}{2}$$

**Example 14.** *Find the equation of the normal at the point 'θ' on the curve $x = 3\cos\theta - \cos^3\theta$, $y = 3\sin\theta - \sin^3\theta$ and show that at the point, where $\theta = \pi/4$, the normal passes through the origin.*

**Sol.** Equations of the curve are $\left.\begin{array}{l} x = 3\cos\theta - \cos^3\theta \\ y = 3\sin\theta - \sin^3\theta \end{array}\right]$ ...(*i*)

$$\frac{dx}{d\theta} = -3\sin\theta - 3\cos^2\theta(-\sin\theta) = -3\sin\theta(1-\cos^2\theta)$$
$$= -3\sin\theta\cdot\sin^2\theta = -3\sin^3\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta - 3\sin^2\theta\cos\theta = 3\cos\theta(1-\sin^2\theta)$$
$$= 3\cos\theta\cdot\cos^2\theta = 3\cos^3\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos^3\theta}{-3\sin^3\theta} = -\cot^3\theta$$

Slope of tangent at 'θ' $= -\cot^3\theta$

∴ Slope of normal at 'θ' $= \tan^3\theta$

Equation of the normal at 'θ' is

$$y - (3\sin\theta - \sin^3\theta) = \tan^3\theta[x - (3\cos\theta - \cos^3\theta)]$$

or $$y - 3\sin\theta + \sin^3\theta = \frac{\sin^3\theta}{\cos^3\theta}[x - 3\cos\theta + \cos^3\theta]$$

or $$\frac{y}{\sin^3\theta} - \frac{3}{\sin^2\theta} + 1 = \frac{x}{\cos^3\theta} - \frac{3}{\cos^2\theta} + 1$$

or $$x\sec^3\theta - y\,\mathrm{cosec}^3\theta = 3\left[\frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}\right]$$
$$= 3(\sec^2\theta - \mathrm{cosec}^2\theta)$$

NOTES

At $\theta = \frac{\pi}{4}$ eqation of normal is

$$x \sec^3 \frac{\pi}{4} - y \operatorname{cosec}^3 \frac{\pi}{4} = 3\left(\sec^2 \frac{\pi}{4} - \operatorname{cosec}^2 \frac{\pi}{4}\right)$$

or $$x(\sqrt{2})^3 - y(\sqrt{2})^3 = 3[(\sqrt{2})^2 - (\sqrt{2})^2] \text{ or } 2\sqrt{2}x - 2\sqrt{2}y = 0$$

or $x - y = 0$ which clearly passes through the origin

## EXERCISE B

1. Find the equation of the tangent and normal to the curve $x = a \cos \theta$, $y = b \sin \theta$ at the point 'θ'.
2. Find the equations of the tangent and normal at $\theta = \frac{\pi}{2}$ to the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$
3. Show that the equation of the tangent to the curve is of the form $x \cos^3 \theta + y \sin^3 \theta = a$.
4. Find the equations of the tangent and the normal on the curve

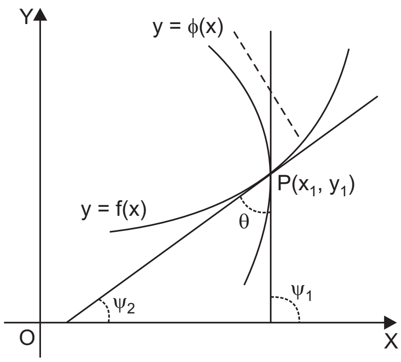
$$x = a \cos^3 \theta,\ y = b \sin^3 \theta \text{ at the point } \theta = \frac{\pi}{4}$$

5. Find the length ofthe portion of the tangent intercepted between the co-ordinate axes at any point of the curve $x = a \cos^3 t$, $y = b \sin^3 t$.
6. Prove that the portion of the tangent to the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at the point θ, intercepted between the axes, is of constant length.

## ANGLE OF INTERSECTION OF TWO CURVES

### Definition

The angle of intersection of two curves is defined ast he angle between the tangents to the two curves at a point of intersection.




Consider the two curves $y = f(x)$ and $y = f(x)$ intersection at $P(x_1, y_1)$.

Let $m_1$ and $m_2$ be the slopes of the tangents at P to the two curves.

If q is the angle between the tangents at P, then

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

In particular, if $m_1 m_2 = -1$ the curves cut orthogonally, and if $m_1 = m_2$ curves touch each other.

**Working Rule:**

1. Solve the equations of the curves simultaneously to find their point (or points) of intersection.

2. Find $\frac{dy}{dx}$ for both the curves separately.

3. Take one of the points of intersection. At this point find the value of $\frac{dy}{dx}$ for both the curves separately. These values gives us slopes of the tangents to the two curves at that point of intersection. Call them $m_1$ and $m_2$.



4. Angle θ between the curves is given by $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

   **Note.** Sometimes, it is convenient to take $(x_1, y_1)$ as a point intersection.

## SOLVED EXAMPLES

**Example 15.** *Find the angle of intersection between the curves.*

(*a*) $x^2 - y^2 = a^2$ *and* $x^2 + y^2 = \sqrt{2}a^2$ (*b*) $y^2 = ax$ *and* $x^2 + y^2 = 2a^2$

(*c*) $x^2 + y^2 = 2a^2$ *and* $xy = a^2$ (*d*) $x^2 + y^2 = 8$ *and* $xy = 4$

(*e*) $y^2 = 4ax$ *and* $x^2 = 4by$

**Sol.** (*a*) Let $(x_1, y_1)$ be a point of intersecstion of the curves

$$x^2 - y^2 = a^2 \quad \ldots(i)$$

and $$x^2 + y^2 = \sqrt{2}a^2 \quad \ldots(ii)$$

Then $$x_1^2 - y_1^2 = a^2 \quad \ldots(iii)$$

$$x_1^2 + y_1^2 = \sqrt{2}a^2 \quad \ldots(iv)$$

Adding $$2x_1^2 = a^2(\sqrt{2} + 1)$$

Subtracting $$2y_1^2 = a^2(\sqrt{2} - 1)$$

Multiplying $$4x_1^2 y_1^2 = a^4(2 - 1) = a^4$$

∴ $$2x_1 y_1 = \pm a^2 \quad \ldots(v)$$

For the curve (*i*), $2x - 2y\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = \frac{x}{y}$

For the curve (*ii*), $2x + 2y\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = \frac{x}{y}$

At $(x_1, y_1)$ slopes of tangents are $\frac{x_1}{y_1}$ and $-\frac{x_1}{y_1}$

∴ $$\tan\theta = \frac{\frac{x_1}{y_1} - \left(-\frac{x_1}{y_1}\right)}{1 + \frac{x_1}{y_1}\left(-\frac{x_1}{y_1}\right)} = \frac{\frac{2x_1}{y_1}}{1 - \frac{x_1^2}{y_1^2}} = \frac{2x_1 y_1}{y_1^2 - x_1^2}$$

$$= \frac{\pm a^2}{-a^2} \quad | \because \text{ of } (v) \text{ and } (iii)$$

$$= \pm 1$$

∴ The angle between the curves = 45°

(*b*) Equations of curves are $y^2 = ax$ ...(*i*)

$$x^2 + y^2 = 2a^2 \quad \text{...}(ii)$$

Eliminating $y$ between (*i*) and (*ii*)

NOTES

$$x^2 + ax - 2a^2 = 0 \qquad (x + 2a)(x - a) = 0$$

$\therefore$ $\quad x = -2a, a$

when $x = -2a$ from (*i*) $\quad y^2 = -2a^2$

which gives imaginary values of $y$ and is rejected

when $x = a$ from (*i*) $\quad y^2 = a^2, y = \pm a$

$\therefore$ The two points of intersections are $(a, a)$ and $(a, -a)$

For the curve (*i*), $\quad 2y\dfrac{dy}{dx} = a \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{a}{2y}$

For the curve (*ii*), $\quad 2x + 2y\dfrac{dy}{dx} = 0 \quad \therefore \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

At $(a, a)$ slopes of tangents are $\dfrac{1}{2}$ and $-1$

$$\tan\theta_1 = \frac{\frac{1}{2} - (-1)}{1 + \frac{1}{2}(-1)} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$$

$\therefore$ $\quad \theta_1 = \tan^{-1}(3)$

At $(a, -a)$ slopes of tangents are $-\dfrac{1}{2}$ and 1

$$\tan\theta_2 = \frac{1 - \left(-\frac{1}{2}\right)}{1 + 1\left(-\frac{1}{2}\right)} = \frac{\frac{3}{2}}{\frac{3}{2}} = 3$$

$\therefore$ $\quad \theta_2 = \tan^{-1}(3)$

(*c*) Equations of curves are $x^2 + y^2 = 2a^2$ ...(*i*)

$$xy = a^2$$

From (*ii*) $\quad y = \dfrac{a^2}{x}$

$\therefore$ From (*i*), $\quad x^2 + \dfrac{a^4}{x^2} = 2a^2, \; x^4 - 2a^2x^2 + a^4 = 0$

$$(x^2 - a^2)^2 = 0, \; x^2 - a^2 = 0 \text{ or } x = \pm a$$

when $\quad x = a$ from (*ii*) $y = a$

when $\quad x = -a$ from (*ii*) $y = -a$

$\therefore$ The two points of intersection are $(a, a)$ and $(-a, -a)$

For the curve (*i*), $\quad 2x + 2y\dfrac{dy}{dx} = 0 \quad \therefore \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

For the curve (*ii*), $\quad y + x\dfrac{dy}{dx} = 0 \quad \therefore \quad \dfrac{dy}{dx} = -\dfrac{y}{x}$

At $(a, a)$ slopes of tangents are $-1$ and $-1$ *i.e.*, equal.

NOTES

∴ **The curves touch each other.**

At $(-a, -a)$ slopes of tangents are $-1$ and $-1$ *i.e.* equal

∴ The curves touch each other.

Hence at each point of intersection, the curves touch each other

(*d*) Please try yourself. It is part (*c*) with $a = 2$

(*e*) Equations of curves are $y^2 = 4ax$ ...(*i*)

$$x^2 = 4by \quad \text{...}(ii)$$

From (*ii*), $$y = \frac{x^2}{4b}$$

∴ From (*i*), $$\frac{x^4}{16b^2} = 4ax \quad \text{or} \quad x(x^3 - 64ab^2) = 0$$

∴ $$x = 0,\ 4a^{1/3}b^{2/3}$$

when $x = 0$ from (*i*)

when $x = 4a^{1/3}b^{2/3}$ from (*ii*)

$$y = \frac{16a^{2/3}b^{4/3}}{4b} = 4a^{2/3}b^{1/3}$$

∴ The points of intersection are (0, 0) and $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

For the curve (*i*), $$2y\frac{dy}{dx} = 4a \quad \therefore \quad \frac{dy}{dx} = \frac{2a}{y}$$

For the curve (*ii*), $$2x = 4b\frac{dy}{dx} \quad \therefore \quad \frac{dy}{dx} = \frac{x}{2b}$$

At $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ slopes of tangents to (1) and (2) are $\frac{a^{1/3}}{2b^{1/3}}$ and $\frac{2a^{1/3}}{b^{1/3}}$

If θ is the angle between them, then

$$\tan\theta = \frac{\frac{2a^{1/3}}{b^{1/3}} - \frac{a^{1/3}}{2b^{1/3}}}{1 + \frac{2a^{1/3}}{b^{1/3}} \cdot \frac{a^{1/3}}{2b^{1/3}}} = \frac{\frac{3}{2} \cdot \frac{a^{1/3}}{b^{1/3}}}{1 + \frac{a^{2/3}}{b^{2/3}}} = \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$$

∴ $$\theta = \tan^{-1}\left[\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}\right]$$

At (0, 0) slopes of tangents are ∞ and 0 *i.e.*, axis of $y$ and $x$ are tangents of two curves. Hence the curves cut each other orthogonally at (0, 0).

**Example 16.** *Show that the curves*

$$\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1 \quad \textit{and} \quad \frac{x^2}{a^2 + \lambda_2} + \frac{y^2}{b^2 + \lambda_2} = 1$$

*can orthogonally.*

**Sol.** Let $$\frac{1}{a^2 + \lambda_1} = l; \quad \frac{1}{b^2 + \lambda_1} = m$$

$$\frac{1}{a^2 + \lambda_2} = l'; \quad \frac{1}{b^2 + \lambda_2} = m'$$

Equations of curves becomes $lx^2 + my^2 = 1$ ...(i)

and $l'x^2 + m'y^2 = 1$ ...(ii)

They cut orhtogonally if

$$\frac{1}{l} - \frac{1}{m} = \frac{1}{l'} - \frac{1}{m'}$$

*i.e.*, if $(a^2 + \lambda_1) - (b^2 + \lambda_1) = (a^2 + \lambda_2) - (b^2 + \lambda_2)$

*i.e.*, if $a^2 - b^2 = a^2 - b^2$ which is true

Hence the curves cut orthogonally.

**Example 17.** *Find the condition that the curves*

$$\frac{x^2}{a} + \frac{y^2}{b} = 1 \text{ and } \frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1 \text{ may cut orthogonally.}$$

**Sol.** Let $\frac{1}{a} = l, \quad \frac{1}{b} = m$

$\frac{1}{\alpha} = l', \quad \frac{1}{\beta} = m'$

Equations of curves are $lx^2 + my^2 = 1$ ...(i)

and $l'^{x2} + m'y^2 = 1$ ...(ii)

They cut orthogonally, if

$$\frac{1}{l} - \frac{1}{m} = \frac{1}{l'} - \frac{1}{m'}$$

or if $a - b = \alpha - \beta$

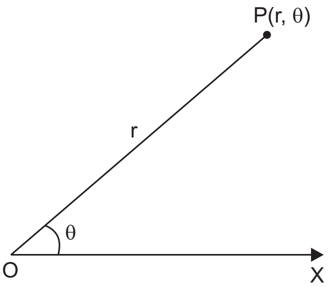
## EXERCISE C

1. Find the angle of intersection between the following curves:
$$y^2 = 2x \text{ and } x^2 + y^2 = 8$$
2. Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$ cut orthogonally.
3. Find the angle of intersection of the curves
$$y^2 = 2ax \text{ and } y^2 = a^2 - x^2$$
4. Show that the curves $lx^2 + my^2 = 1$ and $l'x^2 + m'y^2 = 1$ will intersect orthogonally if
$$\frac{1}{l} - \frac{1}{m} = \frac{1}{l'} - \frac{1}{m'}$$
5. Show that the curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut orthogonally.
6. Show that the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$

   touch if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$.

## POLAR COORDINATES






Let O be a fixed point and OX a fixed straight line through O. The positive direction of OX is indicated by the arrow-head. The fixed point O is called the *pole* or the *origin* and the fixed straight line OX is called the initial line or the polar axis. Let P be any poitn in the plane containing OX. Join OP.

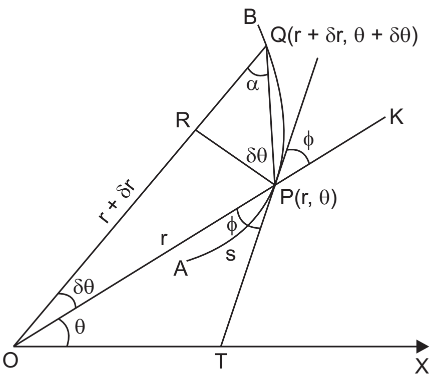
The length OP is called the *radius vector* of the point P and is denoted by '$r$'. The angle XOP is called the *vectorial* angle of the point P and is denoted by '$\theta$'. The numbers $r$ and $\theta$ taken together in this very order are called the *polar coordinates* of the point P and we write it as P($r$, $\theta$).

If ($x$, $y$) are the Cartesian coordinates of P, then

$$x = r\cos\theta, \quad y = r\sin\theta$$

and
$$r = \sqrt{x^2 + y^2}\,, \quad \theta = \tan^{-1}\frac{y}{x}$$

## FOR ANY POINT (r, θ) OF THE CURVE r = f(θ), THE ANGLE φ BETWEEN THE RADIUS VECTOR AND THE TANGENT IS GIVEN BY $\tan\phi = r\frac{d\theta}{dr}$




Let P($r$, $\theta$) be any point on the given curve $r = f(\theta)$ or $f(r, \theta) = 0$.

Let Q($r + \delta r$, $\theta + \delta\theta$) be a point in the neighbourhood of P on the curve.

Join OP, OQ, PQ. Then

$$OP = r, \; OQ = r + \delta r$$

$$\angle XOP = \theta, \; \angle XOQ = \theta + \delta\theta$$

so that
$$\angle POQ = \delta\theta$$

Draw PR $\perp$ OQ.

Let $\angle PQR = \alpha$.

Let the angle between the radius vector OP and the tangent PT

*i.e.,*
$$\angle OPT = \phi$$

From rt. $\angle d\ \Delta$ OPQ,
$$\frac{PR}{OP} = \sin\delta\theta \quad \therefore \quad PR = r\sin\delta\theta$$

$$RQ = OQ - OR = (r + \delta r) - OP\cos\delta\theta = r + \delta r - \cos\delta\theta$$

$$= \delta r + r(1 - \cos\delta\theta) = \delta r + 2r\sin^2\frac{\delta\theta}{2}$$

$\therefore$
$$\tan\alpha = \frac{PR}{RQ} = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\delta\theta/2}$$

NOTES

Dividing the numerator and numerator by $\delta\theta$

$$\tan\alpha = \frac{r\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + r\cdot\dfrac{\sin\delta\theta/2}{\delta\theta/2}\cdot\sin\dfrac{\delta\theta}{2}}$$

Note

Note. When Q → P along the curve $\alpha \to \phi$

[$\because$ ultimately PQ becomes the tangent PT and OQ coincides with OP]

$$\therefore \qquad \tan\phi = \underset{Q\to P}{\text{Lt}}\tan\alpha = \underset{\delta\theta\to 0}{\text{Lt}}\frac{r\dfrac{\sin\delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + r\cdot\dfrac{\sin\delta\theta/2}{\delta\theta/2}\cdot\sin\dfrac{\delta\theta}{2}}$$

$$= \frac{r\cdot 1}{dr/d\theta + r\cdot 1\cdot 0} = \frac{r}{dr/d\theta}$$

Hence $\tan\theta = r\dfrac{d\theta}{dr}$

**Remember.** $\phi$ is the angle between the radius vector and the tangent

Relation between $\theta, \phi, \psi : \psi = \theta + \phi$ ($\because$ $\angle XTP = \angle TOP + \angle OPT$)

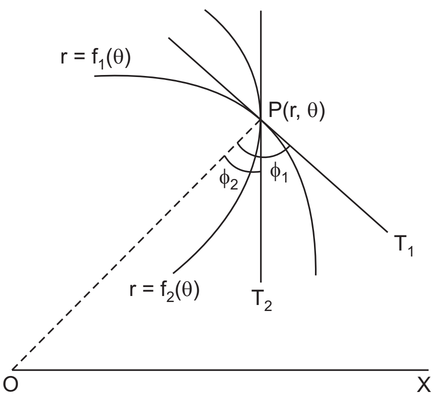
## ANGLE OF ITNERSECTION OF TWO CURVES

Let the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ intersect in P.

Let $\phi_1$, $\phi_2$ be the angles between the common radius vector and OP and the tangents $PT_1$, $PT_2$ to the two curves

Angle of intersection of two curves = angle between their tangents at a point of intersection

$$\therefore \qquad \alpha = \phi_1 - \phi_2 \text{ or } |\ \phi_1 - \phi_2\ |$$




**Cor 1.** For orthogonal intersection

$$\alpha = \frac{\pi}{2} \quad \therefore \quad \phi_1 = \frac{\pi}{2} + \phi_2$$

$$\tan\phi_1 = \tan\left(\frac{\pi}{2} + \phi_2\right) = -\cot\theta_2 = -\frac{1}{\tan\phi_2}$$

or $\tan\phi_1 \cdot \tan\phi_2 = -1$

**Cor. 2.** If $\alpha = 0$, then $\tan \phi_1 = \tan \phi_2$

$\therefore$ The two curves touch if $\tan \phi_1 = \tan \phi_2$.

NOTES

## SOLVED EXAMPLES

**Example 18.** *Find the value of* $\phi$ *for the curve*

$$r^m = a^3 (\cos m\theta - \sin m\theta) \text{ at the point } \theta = 0.$$

**Sol.** Equation of curve is $r^m = a^m (\cos m\theta - \sin m\theta)$

Taking logs, $m \log r = m \log a + \log (\cos m\theta - \sin m\theta)$

Diff. w.r.t.$\theta$,
$$\frac{m}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{-m\sin m\theta - m\cos m\theta}{\cos m\theta - \sin m\theta}$$

or
$$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\frac{\sin m\theta + \cos m\theta}{\cos m\theta - \sin m\theta}$$

$\therefore$
$$\tan \phi = r\frac{d\theta}{dr} = -\frac{\cos m\theta - \sin m\theta}{\sin m\theta + \cos m\theta}$$

At $\theta = 0$,
$$\tan \phi = -\frac{1-0}{0+1} = -1 = \tan\frac{3\pi}{4}$$

Hence
$$\phi = \frac{3\pi}{4}$$

**Example 19.** *Find the angle between the radius vector and the tangent in each of the following curves:*

(*a*) $r = a(1 + \sin\theta)$ *at* $\theta = \pi/6$ (*b*) $r = a \operatorname{cosec}^2 \theta/2$ *at* $\theta = \pi/2$

(*c*) $r = a(1 + \cos\theta)$ *at* $\theta = \pi/2$ (*d*) $r^2 = a^2 \cos 2\theta$ *at* $\theta = \pi/6$

(*e*) $r = a\theta$ (*f*) $r^m = am \cos m\theta$

(*g*) $\dfrac{2a}{r} = 1 - \cos\theta$

**Sol.** (*a*) Equations of curve is $r = a(1+ \sin \theta)$

$$\frac{dr}{d\theta} = a\cos\theta$$

$$\tan\phi = r\frac{d\theta}{dr} = a(1+\sin\theta)\cdot\frac{1}{a\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$$

At
$$\theta = \frac{\pi}{6},\ \tan\phi = \frac{1+\sin\pi/6}{\cos\pi/6} = \frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{2}\cdot\frac{2}{\sqrt{3}} = \sqrt{3}$$

$\therefore$
$$\theta = \frac{\pi}{3}$$

(*b*) Equation of curve is $r = a \operatorname{cosec}^2 \dfrac{\theta}{2}$ ...(*i*)

$$\frac{dr}{d\theta} = a\cdot 2\operatorname{cosec}\frac{\theta}{2}\left(-\operatorname{cosec}\frac{\theta}{2}\cot\frac{\theta}{2}\right)\cdot\frac{1}{2}$$

$$= -a \operatorname{cosec}^2 \frac{\theta}{2} \cot \frac{\theta}{2} = -r \cot \frac{\theta}{2} \qquad | \because \text{ of } (i)$$

NOTES

$$\tan \phi = r\frac{d\theta}{dr} = r \cdot \frac{1}{-r \cot \theta/2} = -\tan \frac{\theta}{2}$$

At $\theta = \frac{\pi}{2}$, $\tan \phi = -\tan \frac{\pi}{4} = \tan \frac{3\pi}{4} \quad \therefore \quad \phi = \frac{3\pi}{4}$

(*c*) Equation of curve is $r = a(1 + \cos \theta)$

$$\therefore \qquad \frac{dr}{d\theta} = -a \sin \theta$$

$$\tan \phi = r\frac{d\theta}{dr} = a(1 + \cos \theta) \cdot \frac{1}{a \sin \theta}$$

$$= -\frac{2 \cos^2 \theta/2}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = -\cot \frac{\theta}{2} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \qquad \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

At $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$

(*d*) Equation of curve is $r^2 = a^2 \cos 2\theta$

Diff., $2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$

$$\frac{dr}{d\theta} = \frac{a^2 \sin 2\theta}{r}$$

$$\tan \phi = \frac{d\theta}{dr} = r\left(-\frac{r}{a^2 \sin 2\theta}\right) = -\frac{r^2}{a^2 \sin 2\theta}$$

$$= -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} \qquad | \because \text{ of } (i)$$

$$= -\cot 2\theta$$

At $\theta = \frac{\pi}{6}$, $\tan \phi = -\cot \frac{\pi}{3} = \tan\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \tan \frac{5\pi}{6}$

$$\therefore \qquad \phi = \frac{5\pi}{6}$$

(*e*) Equation of curve is $r = a\theta$ ...(*i*)

Diff. $\frac{dr}{d\theta} = a$

$$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{a} = \theta \qquad | \because \text{ of } (i)$$

$$\phi = \tan^{-1} \theta$$

(*f*) Equation of curve is $r^m = a^m \cos m\theta$

Taking logarithms, $m \log r = m \log a + \log \cos m\theta$

NOTES

Diff. w.r.t. θ, $\frac{m}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\cos m\theta}\cdot(-m\sin m\theta)$

or $\frac{1}{r}\cdot\frac{dr}{d\theta} = -\tan m\theta$

∴ $\tan\phi = r\frac{d\theta}{dr} - \cot m\theta = \tan\left(\frac{\pi}{2} + m\theta\right)$

⇒ $\phi = \frac{\pi}{2} + m\theta$

(*g*) Equation of curve is $\frac{2a}{r} = 1 - \cos\theta$

Taking logarithms, $\log 2a - \log r = \log(1 - \cos\theta)$

Diff. w.r.t. θ, $0 - \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}(\sin\theta)$

∴ $\tan\phi = r\frac{d\theta}{dr} = \frac{1-\cos\theta}{\sin\theta}$

$= -\frac{2\sin^2\theta/2}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right)$

⇒ $\phi = \pi - \frac{\theta}{2}$

**Example 20.** *Show that the angle between the tangent at any point P and the line joining P to the origin is the same at all pointsof the curve* $\log(x^2+y^2) = k\tan^{-y}/x$.

**Sol.** Equation of curve is $\log(x^2 + y^2) = k\tan^{-1}\frac{y}{x}$

Changing to polar coordinates by putting $x^2 + y^2 = r^2$

and $\tan^{-1}\frac{y}{x} = \theta$, we have

$\log r^2 = k\theta$ or $2\log r = k\theta$

Diff., $\frac{2}{r}\cdot\frac{dr}{d\theta} = k$ ∴ $r\frac{d\theta}{dr} = \frac{2}{k}$

or $\tan\phi = \frac{2}{k}$ ∴ $\phi = \tan^{-1}\frac{2}{k}$

**Example 21.** *Find the angle of intersection of the curves*

$r = a(1 + \cos\theta); r = b(1 - \cos\theta)$

**Sol.** Equations of curves are $r = a(1 + \cos\theta)$ ...(*i*)

and $r = b(1 - \cos\theta)$ ...(*ii*)

For the curve (*i*),

$\frac{dr}{d\theta} = -a\sin\theta$

$r\frac{d\theta}{dr} = a(1+\cos\theta)\cdot\frac{1}{-a\sin\theta}$

$$= -\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} = -\cot\frac{\theta}{2}$$

NOTES

$\therefore \qquad \tan\phi_1 = -\cot\frac{\theta}{2}$

For the curve (*ii*), $\qquad \frac{dr}{d\theta} = -b\sin\theta$

$$r\frac{d\theta}{dr} = b(1-\cos\theta)\cdot\frac{1}{-b\sin\theta}$$

$$= \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2} = \tan\frac{\theta}{2}$$

$\therefore \qquad \tan\phi_2 = \tan\frac{\theta}{2}$

$\therefore \qquad \tan\phi_1\tan\phi_2 = -\cot\frac{\theta}{2}\cdot\tan\frac{\theta}{2} = -1$

Hence the two curves cut orthogonally.

**Example 22.** *Find the angle of intersection of the parabolas*

$$r = \frac{a}{1+\cos\theta} \text{ and } r = \frac{b}{1-\cos\theta}$$

**Sol.** Equations of curves are $r = \frac{a}{1+\cos\theta}$ ...(*i*)

and $\qquad r = \frac{b}{1-\cos\theta}$ ...(*ii*)

*For the curve (i)*

Taking logs,

$$\log r = \log a - \log(1+\cos\theta)$$

Diff. $\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{-\sin\theta}{1+\cos\theta}$

$$= \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$= \tan\frac{\theta}{2}$$

$$\tan\phi_1 = r\frac{d\theta}{dr} = \frac{1}{\tan\theta/2} = \cot\frac{\theta}{2}$$

*For the curve (ii)*

Taking logs,

$$\log r = \log b - \log(1-\cos\theta)$$

Diff., $\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{\sin\theta}{1-\cos\theta}$

$$= -\frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$= -\cot\frac{\theta}{2}$$

$$\tan\phi_2 = r\frac{d\theta}{dr} = -\frac{1}{\cot\theta/2}$$

$\therefore \tan\phi_1\tan\phi_2 = -1$

Hence the two curves cut orthogonally.

**Example 23.** *Show that the circle r = b cuts the curve*

$$r^2 = a^2\cos\theta + b^2 \text{ at an angle } \tan^{-1}\left(\frac{a^2}{b^2}\right)$$

**Sol.** Equations of the curves are $r = b$ ...(*i*)

and $r^2 = a^2 \cos 2\theta + b^2$ ...(*ii*)

Eliminating $r$, $b^2 = a^2 \cos 2\theta + b^2, \cos 2\theta = \cos\frac{\pi}{2}$

or $2\theta = \frac{\pi}{2}$ $\therefore$ $\theta = \frac{\pi}{4}$ Also $r = b$

$\therefore$ Point of intersection is $\left(b, \frac{\pi}{4}\right)$

*For the curve (i)*

$$\frac{dr}{d\theta} = 0$$

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{0} = \infty$$

$$\tan\phi_1 = [\tan\phi]_{\text{at}(b,\ \pi/4)} = \infty$$

$$\therefore \quad \phi_1 = \frac{\pi}{2}$$

*For the curve (ii)*

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

$$\therefore \quad \frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\tan\phi = r\frac{d\theta}{dr} = -\frac{r^2}{a^2 \sin 2\theta}$$

$$\tan\phi_2 = [\tan\phi]_{\text{at }(b,\ \pi/4)} = -\frac{b^2}{a^2}$$

$$\text{or} \quad -\tan\phi_2 = \frac{b^2}{a^2}$$

$$\therefore \quad -\cot\phi_2 = \frac{a^2}{b^2}$$

$$-\tan\left(\frac{\pi}{2} - \phi_2\right) = \frac{a^2}{b^2}$$

$$\tan\left(\phi_2 - \frac{\pi}{2}\right) = \frac{a^2}{b^2}$$

$$\phi_2 - \frac{\pi}{2} = \tan^{-1}\frac{a^2}{b^2}$$

$$\therefore \quad \phi_2 = \frac{\pi}{2} + \tan^{-1}\frac{a^2}{b^2}$$

Angle of intersection of curves $= \phi_2 - \phi_1 = \tan^{-1}\left(\frac{a^2}{b^2}\right)$

## EXERCISE E

**1.** Show that in equiangular spiral $r = ae^{\theta \cot \alpha}$, the tangent is inclined at a constant angle to the radius vector.

**2.** Find the angle at which the radius vector cuts the curve $\frac{1}{r} = 1 + e \cos\theta$.

**NOTES**

3. If *f* is the angle between the tangent to a curve and the radius vector drawn from the region of co-ordinates to the point of contact, prove that $\tan \phi = \dfrac{x \dfrac{dy}{dx}}{x + y \dfrac{dy}{dx}}$.

4. Prove that the tangent at any point $(r, \theta)$ on $r^2 = a^2 \sin 2\theta$ makes an angle $3\theta$ with the initial line.

5. Prove that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n$ intersect orthogonally.

6. Prove that the curves $r^2 = a^2 \cos 2\theta$ and $r = a(1 + \cos \theta)$ intersect and an angle $3 \sin^{-1}\left(\dfrac{3}{4}\right)^{1/4}$.

7. Show that the curves $r^n = a^n \sec (n\theta + \alpha)$ and $r^n = b^n \sec (n\theta + \beta)$ intersect at an angle which is independent of $a$ and $b$.

# 9. ENVELOPES AND EVOLUTES

NOTES

**STRUCTURE**



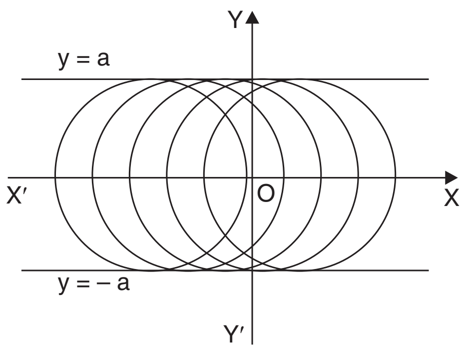
## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- To Find Envelope When Equation of Family of Curves $f(x, y, \alpha) = 0$ is of Form
- Envelope of a Family of Curves $f(x, y, a, b) = 0$, Where Two Parameters $a$, $b$ are Connected by a Relation

## FAMILY OF CURVES

Consider the equation $(x - \alpha)^2 + y^2 = a^2$. It represents a circle of radius $a$ having its centre $(\alpha, 0)$ on the $x$-axis at a distance α from the origin. If we keep $a$ fixed but allow α to take different values, then we get a series of circles each of which is of equal radius $a$, but differs in the position of its centre on $x$-axis.




A system of curves formed in this way is called *a family of curves* and the quantity α different values of which give rise to different members of the family is called the *parameter* of the family of curves. In general, if $f(x, y, \alpha)$ be a function of $x$, $y$ and an *arbitrary constant* α, then the equation $f(x, y, \alpha) = 0$ represents a *family of curves* for different values of α, and α which is constant for the same member of the family but is different for different member curves of the family, is called the **parameter** of the family.

NOTES

A family of curves may also be represented in parametric form as $x = f(t, \alpha)$, $y = \phi(t, \alpha)$, where $\alpha$ is the parameter of the family, and $t$ is a parameter for each member of family.

For example equations $x = \alpha \cos t$, $y = \alpha \sin t$ represent a family of concentric circles $x^2 + y^2 = \alpha^2$. Here $\alpha$ is the parameter of the family of circles, whereas $t$ is a parameter of any particular member of the family.

Let $f(x, y, c) = 0$, where $c$ is a parameter give a family of curves, then the two members of the family $f(x, y, \alpha) = 0$, and $f(x, y, \alpha + \delta\alpha) = 0$ *corresponding to pararmetric values* $\alpha$, *and* $\alpha + \delta\alpha$ of the parameter $c$, are said to be *consecutive* or *contiguous* members of the family. If these two consecutive members cut in a point, then the limiting position of this point of intersection as $\delta\alpha$ tends to zero is called an *ultimate point of intersection* of two consecutive members of the family.

## DEFINITION AND METHOD OF FINDING THE ENVELOPE

**Definition.** *The* **envelope** *of a family of curves is the locus of the limiting position of the points of intersection of any two consecutive members of the family, when one of them tends to coincide with the other, which is kept fixed.*

Let $f(x, y, c) = 0$ be a family of curves, $c$ being the parameter.

Let $f(x, y, \alpha) = 0$ ...(*i*)

and $f(x, y, \alpha + \delta\alpha) = 0$ ...(*ii*)

be two consecutive members of family, corresponding to values $\alpha$, and $\alpha + \delta\alpha$ of parameter $c$.

The envelope will be locus of points of intersection of (*i*) and (*ii*), as $\delta\alpha$ tends to zero.

The points of intersection of two members, satisfy (*i*) and (*ii*) simultaneously and, therefore, also satisfy the equation

$$f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha) = 0$$

Dividing both sides by $\delta\alpha$,

*i.e.*,
$$\frac{f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha)}{\delta\alpha} = 0.$$

Proceeding to limit as $\delta\alpha \to 0$, we see that the limiting positions of points of intersections of (*i*) *and* (*ii*), satisfy the equation $x$

$$\underset{\delta\alpha \to 0}{\text{Lt}} \frac{f(x, y, \alpha + \delta\alpha) - f(x, y, \alpha)}{\delta\alpha} = 0$$

*i.e.*,
$$\frac{\partial f}{\partial \alpha} = 0 \qquad \text{...}(iii)$$

These points being points of intersections of curves (*i*) and (*ii*) also satisfy (*iii*). Hence these limiting positions of points of intersections of curves (*i*) and (*ii*) satisfy eqns. (*i*) and (*iii*) simultaneously.

$\therefore$ **the equation of the envelope is obtained by eliminating $\alpha$ from equations (i) and (ii) i.e., from the equations**

$$\mathbf{f(x, y, \alpha) = 0 \text{ and } \frac{\partial f}{\partial \alpha} = 0.}$$

**Cor.** If we can solve equations, (*i*) and (*iii*) *i.e.*, $f(x, y, \alpha) = 0$, and $\frac{\partial f}{\partial \alpha} = 0$ for $x$ and $y$ in terms of $\alpha$, in the form $x = f(\alpha)$, $y = \phi(\alpha)$, then these two equations are the parametric equations of the envelope, $\alpha$ being the parameter.

## SOLVED EXAMPLES

**Example 1.** *Find the envelope of the families of following curves :*

(*i*) $(x - \alpha)^2 + y^2 = 4\alpha$, *where* $\alpha$ *is the parameter.*

(*ii*) $(x - \alpha)^2 + y^2 = \alpha^2$, *where* $\alpha$ *is the parameter.*

**Sol.** (*i*) Equation of the family of curves is $(x - \alpha)^2 + y^2 = 4\alpha$ ...(1)

Differentiating partially *w.r.t.* the parameter $\alpha$, we get

$$-2(x - \alpha) = 4 \quad \text{or} \quad -2x + 2\alpha = 4$$

or $$2\alpha = 4 + 2x \quad \text{or} \quad \alpha = 2 + x \quad \text{...(2)}$$

Eliminating $\alpha$ between (1) and (2),

[By putting the value of $\alpha$ from (2) in (1)], we get

$$(-2)^2 + y^2 = 4(2 + x) \quad \text{or} \quad 4 + y^2 = 8 + 4x$$

or $$y^2 = 4x + 4 \quad \text{or} \quad y^2 = 4(x + 1)$$

which is the equation of the envelope.

(*ii*) Equation of the family of curves is

$$(x - \alpha)^2 + y^2 = \alpha^2 \quad \text{or} \quad x^2 + \alpha^2 - 2\alpha x + y^2 = \alpha^2$$

or $$f(x, y, \alpha) = x^2 + y^2 - 2\alpha x = 0 \quad \text{...(1)}$$

Differentiating both sides of eqns. (1) partially w.r.t. parameter $\alpha$, we get

$$-2x = 0 \quad \text{or} \quad x = 0 \quad \text{...(2)}$$

It is not possible to eliminate $\alpha$ between eqns. (1) and (2) because eqn. (2) does not contain $\alpha$.

Hence, the given family of curves (1) does not have an envelope.

**Note.** This example 1 (*ii*) part illustrates that *every family of curves need not possess an envelope.* As proved in Art. 2, that the equation of the envelope is obtained by eliminating between the equations

$$f(x, y, \alpha) = 0 \quad \text{...}(i) \qquad \text{and} \qquad \frac{\partial f}{\partial \alpha} = 0 \quad \text{...}(ii)$$

and this eliminant is the condition that equation (*i*) in parameter $\alpha$ may have a **pair of equal roots.** Hence, we conclude that

(*i*) a family of curves in which the parameter occurs only *in the first degree, will not have an envelope.*

(*ii*) if the family of curves is to have an envelope the parameter in it should occur *at least* in the *second degree*.

## TO FIND ENVELOPE WHEN EQUATION OF FAMILY OF CURVES f(x, y, α) = 0 IS A QUADRATIC IN 'α'

Let the quadratic equation $f(x, y, \alpha) = 0$ in parameter $\alpha$, be of the form

$$A\alpha^2 + B\alpha + C = 0 \quad \text{...(1)}$$

where A, B, C are functions of $x$ and $y$.

Differentiating, (1) partially *w.r.t.* $\alpha$, we get

$$2A\alpha + B = 0, \quad \text{or} \quad \alpha = -\frac{B}{2A} \qquad ...(2)$$

NOTES

Eliminating $\alpha$ between (1) and (2) (by putting the value of $\alpha = -\frac{B}{2A}$ from (2) in (1)), we get

$$A\left(\frac{-B}{2A}\right)^2 + B\left(\frac{-B}{2A}\right) + C = 0 \quad \text{or} \quad \frac{B^2}{4A} - \frac{B^2}{2A} + C = 0$$

or $$B^2 - 2B^2 + 4AC = 0 \quad \text{or} \quad -B^2 + 4AC = 0$$

or $$\mathbf{B^2 - 4AC = 0,}$$

**which gives the equation of envelope.**

*Hence, if the equation of a family of curves is a quadratic in the parameter, the equation of its envelope is obtained by simply putting its discriminant equal to zero.*

## TO FIND ENVELOPE WHEN EQUATION OF FAMILY OF CURVES f(x, y, $\alpha$) = 0 IS OF THE FORM A cos $\alpha$ + B sin $\alpha$ = C

Let the equation of the family of curves be

$$A \cos \alpha + B \sin \alpha = C \qquad ...(i)$$

where A, B, C are functions of $x$ and $y$, and $\alpha$ is the parameter.

Differentiating (*i*) *w.r.t.* $\alpha$, we get

$$-A \sin \alpha + B \cos \alpha = 0 \qquad ...(ii)$$

In order to eliminate between (*i*) and (*ii*), square these equations and add. We get

$$A^2(\cos^2 \alpha + \sin^2 \alpha) + B^2(\sin^2 \alpha + \cos^2 \alpha) = C^2$$

or $$A^2 + B^2 = C^2$$

which is the required equation of the envelope.

### SOLVED EXAMPLES

**Example 2**. *Find the envelope of the family of trajectories*

$y = x \tan \alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha}$, $\alpha$ *being the parameter.*

**Sol.** The given equation can be written as

$$y = x \tan \alpha - \frac{gx^2}{2u^2} \sec^2 \alpha = x \tan \alpha - \frac{gx^2}{2u^2}(1 + \tan^2 \alpha)$$

Multiplying by $2u^2$, we get $2u^2y = 2u^2x \tan \alpha - gx^2 - gx^2 \tan^2 \alpha$

$\Rightarrow \quad gx^2 \,.\, \tan^2 \alpha - 2u^2x \,.\, \tan \alpha + (2u^2y + gx^2) = 0.$

This is a quadratic in tan $\alpha$, hence its envelope is given by

Discriminant = 0 *i.e.*, $B^2 - 4AC = 0$ (By Art. 3)

*i.e.*, $$4u^4x^2 - 4gx^2(2u^2y + gx^2) = 0$$

Dividing by $4x^2$, we get $u^4 - g(2u^2y + gx^2) = 0$

or $$u^4 - 2u^2gy - g^2x^2 = 0 \quad \text{or} \quad 2u^2gy + g^2x^2 = u^4.$$

This is the required equation of the envelope.

**Example 3.** *Find the envelope of the family of curves*

$$\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta - c = 0 \text{ for different values of } \theta.$$

**Sol.** Transposing term(s) independent of cos $\theta$ and sin $\theta$ to R.H.S., equation of family of curves is

$$\frac{a^2}{x}\cos\theta - \frac{b^2}{y}\sin\theta = c \quad \text{...(1)}$$

Differentiating partially *w.r.t.* parameter $\theta$, we have

$$-\frac{a^2}{x}\sin\theta - \frac{b^2}{y}\cos\theta = 0 \quad \text{...(2)}$$

Squaring and adding equations (1) and (2), we get

$$\frac{a^4}{x^2}(\cos^2\theta + \sin^2\theta) + \frac{b^4}{y^2}(\sin^2\theta + \cos^2\theta) = c^2$$

or $\frac{a^4}{x^2} + \frac{b^4}{y^2} = c^2$, which is the required equation of the envelope.

**Example 4.** *Find the envelope of the family of curves $x\cos\theta + y\sin\theta = l\sin\theta\cos\theta$ ; $\theta$ being the parameter.*

**Sol.** Equation of the family of curves is

$$x\cos\theta + y\sin\theta = l\sin\theta\cos\theta$$

Dividing every term by sin $\theta$ cos $\theta$, we have

$$\frac{x}{\sin\theta} + \frac{y}{\cos\theta} = l \quad \text{or} \quad x\operatorname{cosec}\theta + y\sec\theta = l \quad \text{...(1)}$$

Diff. both sides of eqn. (1) partially *w.r.t.* $\theta$,

$$-x\operatorname{cosec}\theta\cot\theta + y\sec\theta\tan\theta = 0$$

or $$-x\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta} = -y\cdot\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} \quad \text{or} \quad \frac{x\cos\theta}{\sin^2\theta} = \frac{y\sin\theta}{\cos^2\theta}$$

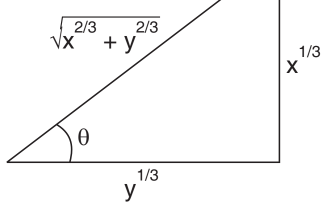
Cross-multiplying $y\sin^3\theta = x\cos^3\theta$

$$\therefore \quad \frac{\sin^3\theta}{\cos^3\theta} = \frac{x}{y} \quad \text{or} \quad \tan^3\theta = \frac{x}{y} \quad \text{...(2)}$$

Let us eliminate $\theta$ from eqns. (1) and (2)

From eqn. (2),

$$\tan\theta = \left(\frac{x}{y}\right)^{1/3} = \frac{x^{1/3}}{y^{1/3}}$$



NOTES

$$\therefore \qquad \sec\theta = \sqrt{\frac{x^{2/3}+y^{2/3}}{y^{1/3}}} \quad \text{and} \quad \operatorname{cosec}\theta = \sqrt{\frac{x^{2/3}+y^{2/3}}{x^{1/3}}}$$

Putting these values of sec θ and cosec θ in eqn. (1),

$$x \cdot \frac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}} + y\frac{\sqrt{x^{2/3}+y^{2/3}}}{y^{1/3}} = l$$

or $$\sqrt{x^{2/3}+y^{2/3}}\left(x^{2/3}+y^{2/3}\right) = l \quad \text{or} \quad (x^{2/3}+y^{2/3})^{3/2} = l$$

Raising both sides to the power $\frac{2}{3}$, $(x^{2/3}+y^{2/3}) = l^{2/3}$

which is the required equation of the envelope of the given family of curves.

## EXERCISE A

**1.** Find the envelope of following families of the curves :

(*i*) $y = mx + \frac{a}{m}$, $m$ being the parameter.

(*ii*) $tx^3 + t^2y = a$, $t$ being the parameter.

(*iii*) $\frac{x^2}{a^2} + \frac{y^2}{k^2-\alpha^2} = 1$, α being the parameter and interpret the result.

[**Hint.** Put $\alpha^2 = t$]

**2.** Find the envelope of the curves

(*i*) $x\cos\alpha + y\sin\alpha = p$ where α is the parameter

(*ii*) $\frac{a^2\cos\theta}{x} - \frac{b^2\sin\theta}{y} = \frac{c^2}{a}$, for different values of θ.

(*iii*) $x\cos\theta + y\sin\theta = a(1+\cos\theta)$ where θ is the parameter.

**3.** Find the envelope of the family of curves

$x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2$, where α is the parameter.

**4.** Find the envelope of ellipse, $x = a\sin(\theta - \alpha)$, $y = b\cos\theta$, where α is the parameter.

**5.** Find the envelope of the family of straight lines

$\frac{ax}{\cos\alpha} - \frac{by}{\sin\alpha} = a^2 - b^2$, α being the parameter.

**6.** Find the envelope of the family of lines $ax\cos\theta + by\cot\theta = a^2 + b^2$, θ being the parameter.

**7.** Find the envelope of the family of curves

$\frac{x^3}{a^3\cos\theta} + \frac{y^3}{b^3\sin\theta} = 1$, θ being the parameter.

**8.** Find the envelope of the family of curves $y = t^2(x-t)$, $t$ being the parameter.

### Answers

**1.** (*i*) $y^2 = 4ax$ (*ii*) $x^6 + 4ay = 0$

(*iii*) $(x^2 - y^2 + k^2)^2 - 4k^2x^2 = 0$. This equation represents a square, the equations of whose four sides are $x \pm y + k = 0$ and $x \pm y - k = 0$.

**2.** (*i*) $x^2 + y^2 = p^2$ (*ii*) $\frac{a^4}{x^2} + \frac{b^4}{y^2} = \frac{c^4}{a^2}$ (*iii*) $x^2 + y^2 = 2ax$

**3.** $4a^2(x^2 + y^2) = (x^2 + y^2 - c^2)^2$ **4.** $x^2 = a^2$ **5.** $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

**6.** $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$ **7.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ **8.** $4x^3 = 27y$.

NOTES

## ENVELOPE OF A FAMILY OF CURVES f(x, y, a, b) = 0, WHERE TWO PARAMETERS a, b ARE CONNECTED BY A RELATION

Let the equation of the family of curves be

$$f(x, y, a, b) = 0 \qquad ...(i)$$

where two parameters $a$ and $b$ are connected by the relation

$$\phi(a, b) = 0 \qquad ...(ii)$$

If convenient eliminate one of parameters $a$ or $b$ between (*i*) and (*ii*). The resulting equation now contains only one parameter and its envelope can be found by the method already explained in article 2.

But if it be not convenient to eliminate one parameter (which generally will be the case) we may regard one of the parameters say $b$ as function of the other parameter $a$, and then differentiating (*i*) and (*ii*) partially *w.r.t.* $a$, we get

$$\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b} \cdot \frac{db}{da} = 0 \quad ...(i) \qquad \text{and} \qquad \frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot \frac{db}{da} = 0 \quad ...(ii)$$

($\because$ By formula of Implicit Differentiation, differentiating $f(x, y) = c$, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = 0)$$

Elimination now of $a$, $b$, $\dfrac{db}{da}$ between the four equations (*i*), (*ii*), (*iii*) and (*iv*), gives the equation of the envelope.

### SOLVED EXAMPLES

**Example 5.** *Find the envelope of the family of lines*

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(i)$$

*where a and b are connected by the relation* $a^n + b^n = c$ ...(*ii*)
*and c is a constant.*

**Sol.** Differentiating (*i*) and (*ii*) partially *w.r.t.* $a$ regarding $b$ as a function of $a$, we have

$$-\frac{x}{a^2} - \frac{y}{b^2} \cdot \frac{db}{da} = 0 \quad ...(iii) \qquad \text{and} \qquad na^{n-1} + nb^{n-1} \cdot \frac{db}{da} = 0 \quad ...(iv)$$

From (*iii*), $\dfrac{db}{da} = -\dfrac{b^2 x}{a^2 y}$. Putting this value of $\dfrac{db}{da}$ in (*iv*), we get

$$na^{n-1} + nb^{n-1}\left(\frac{-b^2 x}{a^2 y}\right) = 0 \quad \text{or} \quad na^{n-1} = \frac{nb^{n+1} x}{a^2 y}$$

NOTES

Dividing by $n$ and cross-multiplying

$$a^{n+1} y = b^{n+1} x \quad \text{or} \quad \frac{x}{a^{n+1}} = \frac{y}{b^{n+1}} \qquad ...(v)$$

Now equation of envelope is obtained by eliminating $a$ and $b$ between ($i$), ($ii$) and ($v$)

From ($v$), we have

$$\frac{\frac{x}{a}}{a^n} = \frac{\frac{y}{b}}{b^n}$$

$$\therefore \quad \frac{\frac{x}{a}}{a^n} = \frac{\frac{y}{b}}{b^n} = \frac{\frac{x}{a} + \frac{y}{b}}{a^n + b^n} \qquad \left[\because \text{ If } \frac{a}{b} = \frac{c}{d}; \text{ then each fraction} = \frac{a+c}{b+d}\right]$$

$$= \frac{1}{c} \qquad [\because \text{ of } (i) \text{ and } (ii)]$$

$$\therefore \quad \frac{x}{a^{n+1}} = \frac{1}{c} \quad \text{and} \quad \frac{y}{b^{n+1}} = \frac{1}{c}$$

$$\therefore \quad a^{n+1} = cx \quad \text{and} \quad b^{n+1} = cy$$

$$\therefore \quad a = (cx)^{\frac{1}{n+1}} \quad \text{and} \quad b = (cy)^{\frac{1}{n+1}}$$

Substituting these values of $a$ and $b$ in ($ii$), we get

$$(cx)^{\frac{n}{n+1}} + (cy)^{\frac{n}{n+1}} = c \quad \text{or} \quad c^{\frac{n}{n+1}} x^{\frac{n}{n+1}} + c^{\frac{n}{n+1}} y^{\frac{n}{n+1}} = c$$

Dividing both sides by $c^{\frac{n}{n+1}}$, we have

$$x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{1-\frac{n}{n+1}} = c^{\frac{1}{n+1}}$$

which is the equation of the envelope.

**Example 6.** *Find the equation of the envelope of the family of curves*

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1 \qquad ...(1)$$

*where a and b are connected by relation* $a^p + b^p = c^p$ ...(2)

**Sol.** Regarding $b$ as a function of $a$, and differentiating (1) and (2) partially $w.r.t.\ a$, we have

$$\frac{-mx^m}{a^{m+1}} - \frac{my^m}{b^{m+1}} \cdot \frac{db}{da} = 0 \quad ...(3) \qquad \text{and} \qquad pa^{p-1} + pb^{p-1}\frac{db}{da} = 0 \quad ...(4)$$

Let us eliminate $\frac{db}{da}$ from equations (3) and (4).

From eqn. (3) $\quad \frac{-my^m}{b^{m+1}} \frac{db}{da} = \frac{mx^m}{a^{m+1}}$

$$\therefore \quad \frac{db}{da} = -\frac{x^m\, b^{m+1}}{y^m\, a^{m+1}}$$

Putting this value of $\frac{db}{da}$ in eqn. (4), we have

$$pa^{p-1} - pb^{p-1}\frac{x^m b^{m+1}}{y^m a^{m+1}} = 0$$

Dividing by $p$ and multiplying by $y^m a^{m+1}$, we have

$$y^m a^{m+p} = x^m b^{m+p}$$

or $$\frac{x^m}{a^{m+p}} = \frac{y^m}{b^{m+p}} \quad \text{or} \quad \frac{\frac{x^m}{a^m}}{a^p} = \frac{\frac{y^m}{b^m}}{b^p} = \frac{\frac{x^m}{a^m}+\frac{y^m}{b^m}}{a^p+b^p}$$

or $$\frac{x^m}{a^{m+p}} = \frac{y^m}{b^{m+p}} = \frac{1}{c^p} \qquad [\because \text{ of (1) and (2)}]$$

From first and third members $a^{m+p} = c^p x^m$

$$\therefore \qquad a = \left(c^p x^m\right)^{\frac{1}{m+p}}$$

From second and third members $b^{m+p} = c^p y^m$

$$\therefore \qquad b = \left(c^p y^m\right)^{\frac{1}{m+p}}$$

Putting these values of $a$ and $b$ in eqn. (2), we have

$$\left(c^p x^m\right)^{\frac{p}{m+p}} + \left(c^p y^m\right)^{\frac{p}{m+p}} = c^p$$

or $$c^{\frac{p^2}{m+p}} x^{\frac{mp}{m+p}} + c^{\frac{p^2}{m+p}} y^{\frac{mp}{m+p}} = c^p$$

Dividing every term by $c^{\frac{p^2}{m+p}}$, we have

$$x^{\frac{mp}{m+p}} + y^{\frac{mp}{m+p}} = c^{\frac{mp}{m+p}} \qquad \left[\because \; p - \frac{p^2}{m+p} = \frac{pm}{m+p}\right]$$

which is the required equation of the envelope.

## EXERCISE B

**1.** Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where parameters $a$ and $b$ are connected by the relation

(*i*) $a^n + b^n = c^n$ (*ii*) $a + b = c$ (*iii*) $a^2 + b^2 = c^2$

(*iv*) $a^3 + b^3 = c^3$ (*v*) $a^m b^n = c^{m+n}$ (*vi*) $ab = c^2$, $c$ being a constant.

**2.** Find the envelope of the family of parabolas $\left[\frac{x}{a}\right]^{1/2} + \left[\frac{y}{b}\right]^{1/2} = 1$, when

(*i*) $a^n + b^n = c^n$ (*ii*) $a + b = c$, $c$ being a constant.

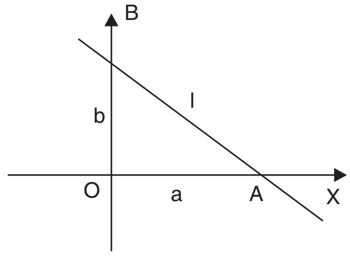
**3.** Find the envelope of curves, $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$, when

(*i*) $a + b = c$, and (*ii*) $ab = c^2$, $c$ being a constant.

NOTES

**4.** Show that the envelope of the straight line of given length $l$, which slides with extremities on two fixed straight lines at right angles is $x^{2/3} + y^{2/3} = l^{2/3}$.

[**Hint.** Let us take the two fixed straight lines at right angles as axes. Let the equation of the line with its ends on the given ⊥ lines as axes be $\frac{x}{a}+\frac{y}{b}=1$ where $a^2 + b^2 = l^2$.]




**Answers**

**1.** (*i*) $x^{n/n+1} + y^{n/n+1} = c^{n/n+1}$ (*ii*) $x^{1/2} + y^{1/2} = c^{1/2}$

(*iii*) $x^{2/3} + y^{2/3} = c^{2/3}$ (*iv*) $x^{3/4} + y^{3/4} = c^{3/4}$

(*v*) $(m + n)^{m+n}\, x^m\, y^n = m^m\, n^n\, c^{m+n}$

(*vi*) $4xy = c^2$

**2.** (*i*) $x^{n/2n+1} + y^{n/2n+1} = c^{n/2n+1}$ (*ii*) $x^{1/3} + y^{1/3} = c^{1/3}$

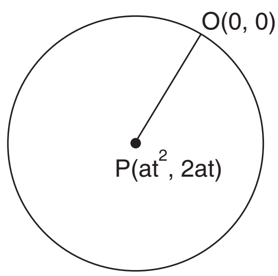
**3.** (*i*) $x^{m/m+1} + y^{m/m+1} = c^{m/m+1}$ (*ii*) $4xy = c^2$.

## SOME MORE ILLUSTRATIVE EXAMPLES

**Example 7.** *Find the envelope of the circles whose centres lie on the parabola and which pass through its vertex.*

**Sol.** Let the equation of the parabola be $y^2 = 4ax$ ...(1)

We that any point on this parabola is P($at^2$, $2at$) and vertex of this parabola is O(0, 0).




Centres of the circles are points of parabola (1) *i.e.*, centre is P($at^2$, $2at$).

Since the circles pass through the vertex O(0, 0),

therefore radius = Distance OP

$$= \sqrt{(at^2-0)^2+(2at-0)^2} = \sqrt{a^2t^4+4a^2t^2}$$

∴ Equation of the circles is $(x - \alpha)^2 + (y - \beta)^2 = r^2$

*i.e.*, $$(x - at^2)^2 + (y - 2at)^2 = a^2t^4 + 4a^2t^2$$

or $$x^2 + a^2t^4 - 2axt^2 + y^2 + 4a^2t^2 - 4ayt = a^2t^4 + 4a^2t^2$$

or $$x^2 - 2axt^2 + y^2 - 4ayt = 0$$

or $$-2axt^2 - 4ayt + x^2 + y^2 = 0$$

Dividing by – 1, $$2axt^2 + 4ayt - (x^2 + y^2) = 0$$

which is a quadratic in parameter $t$.

∴ Equation of the envelope of these circles is $B^2 - 4AC = 0$

or $$16a^2y^2 + 8ax(x^2 + y^2) = 0$$

Dividing every term by $8a$,

$$2ay^2 + x^3 + xy^2 = 0 \quad \text{or} \quad x^3 + y^2(x + 2a) = 0$$

**Example 8.** *Show that the envelope of a family of parabolas* $\left(\frac{x}{a}\right)^{1/2}+\left(\frac{y}{b}\right)^{1/2}=1$ *under the condition* $ab = c^2$ *is a hyperbola having its asymptotes coinciding with the axes.*

**Sol.** Equation of the hyperbolas is

$$\left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{b}\right)^{1/2} = 1 \qquad \ldots(1)$$

where $ab = c^2$ ...(2)

NOTES

Differentiating both equations partially *w.r.t.* $a$, regarding $b$ as a function of $a$,

$$x^{1/2} \frac{d}{da} a^{-1/2} + y^{1/2} \frac{d}{da} b^{-1/2} = 0$$

or $$x^{1/2}\left(-\frac{1}{2}\right)a^{-3/2} + y^{1/2}\left(-\frac{1}{2}\right)b^{-3/2}\frac{db}{da} = 0 \qquad \ldots(3)$$

and $$a\frac{db}{da} + b\,.1 = 0 \qquad \ldots(4)$$

From (4) , $$\frac{db}{da} = -\frac{b}{a}$$

Putting this value of $\frac{db}{da}$ in eqn. (3),

$$-\frac{1}{2}x^{1/2}a^{-3/2} + \frac{1}{2}y^{1/2}b^{-3/2}\frac{b}{a} = 0$$

Dividing by $-\frac{1}{2}$ and multiplying by $a$,

$$x^{1/2}a^{-1/2} - y^{1/2}b^{-1/2} = 0$$

or $$\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 0 \qquad \text{or} \qquad \sqrt{\frac{x}{a}} = \sqrt{\frac{y}{b}}$$

or $$\frac{\sqrt{\frac{x}{a}}}{1} = \frac{\sqrt{\frac{y}{b}}}{1} = \frac{\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}}}{1+1} \qquad \text{or} \qquad \sqrt{\frac{x}{a}} = \sqrt{\frac{y}{b}} = \frac{1}{2} \qquad \text{[By (1)]}$$

$\therefore$ $$\sqrt{\frac{x}{a}} = \frac{1}{2} \qquad \text{and} \qquad \sqrt{\frac{y}{b}} = \frac{1}{2}$$

Squaring $$\frac{x}{a} = \frac{1}{4} \qquad \text{and} \qquad \frac{y}{b} = \frac{1}{4}$$

$\therefore$ $$a = 4x \qquad \text{and} \qquad b = 4y$$

Putting these values of $a$ and $b$ in eqn. (2), we have

$$4x\,.\,4y = c^2 \quad \text{or} \quad 16xy = c^2$$

which we know is a rectangular Hyperbola with asymptotes as axes.

## EXERCISE C

1. Find the envelope of the circles, whose diameter is the radius vactor of the parabola $y^2 = 4ax$.
2. Show that the envelope of the family of circles whose diameters are the double ordinates of the parabola $y^2 = 4ax$ is the parabola $y^2 = 4a(x + a)$.
3. Find the envelope of the circles drawn upon the central radii vectors of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as diameter.

**4.** Find the envelope of the circles which pass through the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are their centres are upon its circumference.

NOTES

**5.** Show that the envelope of the polars of the points on the ellipse $\frac{x^2}{h^2}+\frac{y^2}{k^2}=1$ w.r.t. ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \frac{h^2x^2}{a^4}+\frac{k^2y^2}{b^4}=1.$$

[**Hint.** We know that any point on the ellipse $\frac{x^2}{h^2}+\frac{y^2}{k^2}=1$ is $(h\cos\theta, k\sin\theta)$.

We also know that equation of the polar of the point $(h\cos\theta, k\sin\theta)$ *w.r.t* the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x\,.\,h\cos\theta}{a^2}+\frac{y\,.\,k\sin\theta}{b^2}=1$].

**6.** Show that the envelope of the straight lines joining the extremities of a pair of semi-conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}$.

[**Hint.** We know that the extremities of a pair of semi-conjugate diameters of the ellipse are $(a\cos\theta, b\sin\theta)$, $\left[a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right]=(-a\sin\theta, b\cos\theta)$.]

**7.** (*a*) Find the envelope of the ellipses having the axes of co-ordinates as principal axes and the semi-axes $a$ and $b$ connected by the relation $ab = c^2$.

(*b*) Find the envelope of a system of concentric and co-axial ellipses of constant area.

[**Hint.** Area of the ellipse = $\pi ab$.]

**8.** Find the envelope of the ellipses having the co-ordinate axes as principal axes and sum of their semi-axes is constant.

[**Hint.** $a + b = c$.]

**9.** Find the envelope of the family of ellipses such that $a^2 + b^2 = c$.

*OR*

Find the envelope of the ellipses having the axes of co-ordinates as principal axes and the sum of the squares of their semi-axes constant.

**10.** Find the envelope of the straight lines drawn through the extremities of and at right angles to the radii vectors of the curve $r = a(1 + \cos\theta)$.

**11.** Show that the envelope of the family of curves $A\lambda^3 + 3B\lambda^2 + 3C\lambda + D = 0$ where $\lambda$ is the parameter, and A, B, C, D are functions of $x$ and $y$ is $(BC - AD)^2 = 4(BD - C^2)(AC - B^2)$.

**12.** Show that the radius of curvature of the envelope of the line

$$x\cos\alpha + y\sin\alpha = f(\alpha) \text{ is } f(\alpha) + f''(\alpha).$$

## Answers

**1.** $ay^2 + x(x^2 + y^2) = 0$

**3.** $a^2x^2 + b^2y^2 = (x^2 + y^2)^2$

**4.** $4(a^2x^2 + b^2y^2) = (x^2 + y^2)^2$

**7.** (*a*) $2xy = c^2$ (*b*) $xy = k$ where $k$ is a constant

**8.** $x^{2/3} + y^{2/3} = c^{2/3}$

**9.** $x + y = \sqrt{c}$

**10.** $r = 2a\cos\theta$.

NOTES

## GEOMETRICAL RELATION BETWEEN A FAMILY OF CURVES AND ITS EVOLUTE

*To show that in general, the envelope of a family of curves, touches each member of the family.*

Let the family of curves be given by the equation

$$F(x, y, \alpha) = 0 \quad ...(1)$$

where $\alpha$ is the parameter.

We know that equation of its envelope is found by eliminating the parameter $\alpha$ between (1) and $\frac{\partial F}{\partial \alpha} = 0$ ...(2)

On solving (1) and (2) as simultaneous equations for $x$ and $y$ in terms of $\alpha$,

Let
$$\left.\begin{aligned} x &= f(\alpha) \\ y &= \phi(\alpha) \end{aligned}\right\} \quad ...(3)$$

be the parametric equations of the envelope, $\alpha$ being the parameter.

We know that slope of the tangent to envelope (3) is

$$\frac{dy}{dx} = \frac{dy/d\alpha}{dx/d\alpha} = \frac{\phi'(\alpha)}{f'(\alpha)} \quad ...(4)$$

Of course, the equations (3) will satisfy equation (1) for every value of $\alpha$.

Differentiating (1) *w.r.t.* $\alpha$, regarding $x$, $y$ as functions of $\alpha$, we get

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{d\alpha} + \frac{\partial F}{\partial y} \cdot \frac{dy}{d\alpha} + \frac{\partial F}{\partial \alpha} = 0 \quad ...(5)$$

Putting $\frac{\partial F}{\partial \alpha} = 0$ from eqn. (2) in eqn. (5), we have

$$\frac{\partial F}{\partial x} \cdot f'(\alpha) + \frac{\partial F}{\partial y} \cdot \phi'(\alpha) = 0 \quad \text{or} \quad \frac{\phi'(\alpha)}{f'(\alpha)} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y} \quad ...(6)$$

Now, R.H.S. of (6) *i.e.*, $\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \left( = \frac{dy}{dx} \right)$ is the slope of the tangent at an ordinary point $(x, y)$ of the curve '$\alpha$' of the family (1), where as L.H.S. of (6) is the slope of tangent at the same point to the envelope (*iii*) [By eqn. (4)]. These slopes being equal, the envelope and curve '$\propto$' of family have the same tangent at the common point, and hence they touch each other.

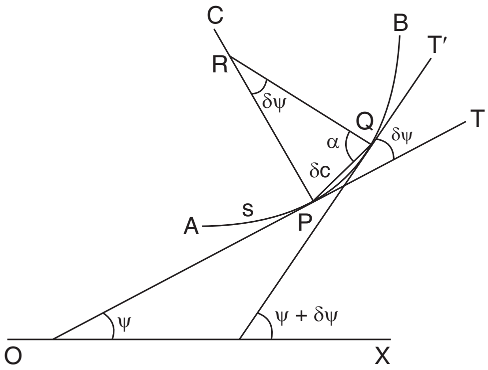
**Note.** If at any common point of envelope and the curve ;

$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$, the above argument fails, therefore, the envelope may not touch the curve at its *singular points* (points where both $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ vanish simultaneously).

## EVOLUTE AS ENVELOPE OF NORMALS

NOTES

*To prove that the evolute of a curve is the envelope of its normals.*




Let PT and QT′ be tangents to the curve AB, at two neighbouring points P and Q, and let the normals at P and Q intersect in R, then from figure, it is clear that ∠PRQ = $\delta\psi$.

[∵ Angle between two tangents = Angle between their normals]

Let $s$ be length of arc AP and $s + \delta s$, be length of arc AQ, so that arc PQ = $\delta s$.

Let chord PQ = $\delta c$ and ∠PQR = $\alpha$.

Applying Sine formula to the triangle PQR, we have

$$\frac{\text{PR}}{\sin \angle \text{PQR}} = \frac{\text{chord PQ}}{\sin \angle \text{PRQ}} \quad \textit{i.e.,} \quad \frac{\text{PR}}{\sin \alpha} = \frac{\delta c}{\sin \delta\psi}$$

$$\therefore \qquad \text{PR} = \sin \alpha \frac{\delta c}{\sin \delta\psi} = \sin \alpha \, . \, \frac{\delta c}{\delta s} \, . \, \frac{\delta s}{\delta \psi} \, . \, \frac{\delta\psi}{\sin \delta\psi}$$

Now Let Q → P along the curve, then chord PQ becomes tangent at P and normal QR becomes PR and hence $\alpha \to \pi/2$

$$\therefore \qquad \underset{\text{Q} \to \text{P}}{\text{Lt}} \text{PR} = \sin \frac{\pi}{2} \, . \, 1 \, . \, \rho \, . \, 1$$

$$\left[ \because \; \frac{\delta c}{\delta s} \to 1 \text{ and } \frac{\delta\psi}{\sin \delta\psi} \to 1 \text{ and } \text{Lt} \frac{\delta s}{\delta \psi} = \rho, \text{Radius of curvature} \right]$$

$= \rho =$ PC where $c$ is the centre of curvature.

Corresponding to point P of the curve.

Thus the limiting position of R is the point C ...(1)

We know by definition of evolute of a curve that locus of C, for different positions of P on the curve is the evolute of the curve. But C, also being the ultimate point of intersection of any two consecutive normals, its locus is the envelope of the normals to the curve. (By Def. of Envelope). Hence evolute of a curve is the envelope of its normals.

**Note.** The above theorem gives us another "*definition of the evolute of a curve, as the envelope of its normals*," and thereby suggests an alternative method of finding the evolute of a curve as the envelope of its normals.

### SOLVED EXAMPLES

**Example.** *Find the evolute of the parabola* $y^2 = lx$ *regarding it as the envelope of its normals.*

**Sol.** Equation of the parabola is $y^2 = lx$ ...(1)

Comparing with $y^2 = 4ax$, we have

$$4a = l \qquad \therefore \qquad a = \frac{l}{4}$$

We know that equation of the normal at the point $(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$\mathbf{tx + y = 2at + at^3}$$

NOTES

Putting $a = \frac{l}{4}$, equation of normal is $tx + y = \frac{l}{2}t + \frac{l}{4}t^3$

or $$\frac{l}{4}t^3 + \left(\frac{l}{2} - x\right)t - y = 0 \qquad ...(2)$$

We know by Art. 7, that evolute of parabola (1) is the envelope of normals (2) to parabola (1).

Differentiating both sides of eqn. (2) partially *w.r.t.* $t$

$$\frac{l}{4}.3t^2 + \left(\frac{l}{2} - x\right) = 0 \quad \text{or} \quad \frac{3}{4}lt^2 = x - \frac{l}{2}$$

$\therefore$ $$t^2 = \frac{4}{3l}\left(\frac{2x - l}{2}\right) = 2\left(\frac{2x - l}{3l}\right) \qquad ...(3)$$

Let us eliminate $t$ from eqns. (2) and (3)

From eqn. (2) $$t\left[\frac{l}{4}t^2 + \frac{l}{2} - x\right] = y$$

Squaring both sides $$t^2\left[\frac{lt^2 + 2l - 4x}{4}\right]^2 = y^2$$

Putting the value of $t^2$ from (3),

$$2\left(\frac{2x - l}{3l}\right)\frac{\left[\frac{2}{3}(2x - l) + 2l - 4x\right]^2}{16} = y^2$$

or $$\left(\frac{2x - l}{24}\right)\frac{(4x - 2l + 6l - 12x)^2}{9} = y^2$$

or $$(2x - l)\,(-8x + 4l)^2 = 24 \times 9y^2$$

or $$16(2x - l)\,(2x - l)^2 = 216y^2$$

Dividing by 8, $$2(2x - l)^3 = 27y^2$$

which is the required equation of envelope of normals (2) to parabola (1) or evolute of parabola (1).

## EXERCISE D

1. Define envelope. Show that the evolute of a curve is the envelope of its normals.
[**Hint.** It is Art. 7.]

2. Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ regarding it as envelope of normals.

[**Hint.** Equation of normal to the ellipse at $(a\cos\theta, b\sin\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.]

3. Find the equation of any normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Hence find its evolute.

**NOTES**

**4.** Prove that the equation of the normal to curve $x^{2/3} + y^{2/3} = a^{2/3}$, may be written in the form $x \sin \phi - y \cos \phi + a \cos 2\phi = 0$, and hence deduce the equation of its evolute.

**5.** From any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, perpendiculars are drawn to the axes. Show that the line joining the feet of the perpendiculars always touches the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1 .$$

[**Hint.** Use the result of Art. 6.]

### Answers

**2.** $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

**3.** $ax \cos \theta + by \cot \theta = a^2 + b^2$, $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$

**4.** $(x + y)^{2/3} + (x - y)^{2/3} = 2\, a^{2/3}$.

# 10. CURVATURE

## STRUCTURE

Introduction
Definition
Curvature of Circle
Radius of Curvature for Cartesian Equation
Convention of Signs
Note on Parabola $y^2 = 4ax$
Note on Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Let us Find ρ for Curve (1) using Equations (2)
Radius of Curvature for Polar Equations
Radius of Curvature for Pedal Equations $\left(\text{To prove } \rho = r\frac{dr}{dp}\right)$
Radius of Curvature for Tangential Polar Equations $p = f(\psi)$
$\left(\text{To prove } \rho = r\frac{dr}{dp}\right)$
Radius of Curvature at the Origin
Centre of Curvature, Circle of Curvature and Evolute
Chord of Curvature

## LEARNING OBJECTIVES

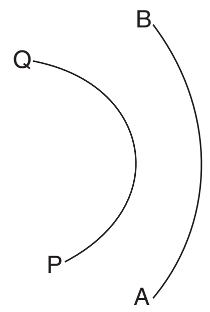
*After going through this unit you will be able to:*

- Let us Find ρ for Curve (1) using Equations (2)
- Radius of Curvature for Polar Equations
- Radius of Curvature at the Origin
- Centre of Curvature, Circle of Curvature and Evolute
- Chord of Curvature

## INTRODUCTION

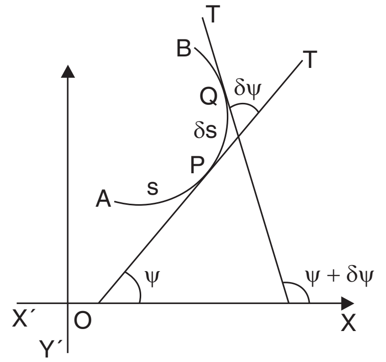
NOTES

In the adjoining figure, curve PQ bends more sharply than the curve AB. The measure of the sharpness of the bending of a curve at a particular point is called *curvature* of the curve at the point. In this chapter, we shall find mathematical expressions for the curvature of a curve at a given point which will give a definite numerical measure of bending of the curve at that point.




## DEFINITIONS

Let P, Q be two neighbouring points on a curve AB. Let arc AP = $s$, arc AQ = $s + \delta s$, so that arc PQ = $\delta s$, A being a fixed point on the curve, from which arcs are measured. Let the tangents to the curve at points P and Q make angles $\psi$ and $\psi + \delta\psi$ respectively with a fixed line *say* $x$-axis. Then




(*i*) the *angle* $\delta\psi$ through which the tangent turns as its point of contact travels along the arc PQ is called the **total bending** or **total curvature** *of arc PQ* ;

(*ii*) *the ratio* $\frac{\delta\psi}{\delta s}$ *is called the* **mean** *or* **average curvature** of *arc* PQ ;

(*iii*) *the limiting value of the mean curvature when* $Q \to P$ *is called the* **curvature of the curve at the point P.** *Thus, the curvature* (*k*) *at point*

$$P = \underset{Q \to P}{Lt} \frac{\delta\psi}{\delta s} = \underset{\delta s \to 0}{Lt} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}; \text{ and}$$

(*iv*) *the reciprocal of the curvature of the curve at P, provided this curvature is not zero, is called the* **radius of curvature of the curve at P.** This is usually denoted by $\rho$. Thus,

$$\rho = \frac{1}{k} = \frac{ds}{d\psi}.$$

## CURVATURE OF CIRCLE

*To show that the curvature of a circle is constant and equal to the reciprocal of its radius.*

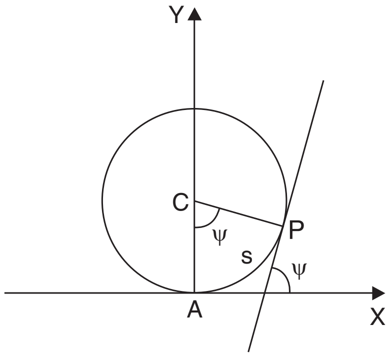
Consider any circle with centre C and radius $r$.

Let A, the lowest point of the circle be taken as the (fixed point) origin and tangent at A as $x$-axis.

Let P be any point on the circle such that arc AP = $s$ and tangent at P make an angle $\psi$ with $x$-axis.

$\therefore \quad \angle ACP = \psi \quad$ [$\because$ Angle between two lines
= Angle between their perpendiculars]

$\therefore \quad s = r\psi \quad$ [$\because$ From trigonometry $l = r\theta$ for a circle]




NOTES

Differentiating w.r.t. $\psi$, $\dfrac{ds}{d\psi} = r$

$\therefore \quad \text{Curvature} = \dfrac{d\psi}{ds} = \dfrac{1}{r}$

*i.e.,* curvature at *every* point of the circle is reciprocal of its radius and hence is constant.

**Note 1.** $\because$ Curvature at any point of the circle $= \dfrac{1}{r}$

$\therefore$ As $r$ increases, curvature decreases

*i.e.,* Curvature becomes smaller and smaller as the radius of circle becomes larger and larger.

**2.** $\because$ Curvature $= \dfrac{1}{r}$

$\therefore$ Radius of curvature $= r =$ Radius of circle.

**Remark.** Conversely, if curvature of a curve is constant ;

*i.e.,*
$$\frac{d\psi}{ds} = \frac{1}{r} \text{ (say)} \quad \therefore \quad d\psi = \frac{1}{r}\, ds$$

Integrating both sides, $\psi = \dfrac{1}{r} s \quad$ or $\quad s = r\psi$

$\therefore$ The curve is a circle.

$\therefore$ *"The circle is the only curve of constant curvature".*

*Let us find the radius of curvature at the point* $(s, \psi)$ *on the curve* $s = a \log(\tan\psi + \sec\psi) + a\tan\psi\sec\psi$.

$$\rho = \frac{ds}{d\psi} = a\frac{1}{\tan\psi + \sec\psi}(\sec^2\psi + \sec\psi\tan\psi)$$
$$+ a(\sec\psi\sec^2\psi + \tan\psi\sec\psi\tan\psi)$$
$$= a\sec\psi + a\sec\psi(\sec^2\psi + \tan^2\psi)$$
$$= a\sec\psi\,.\,[1 + \tan^2\psi + \sec^2\psi] = 2a\sec^3\psi.$$

The relation between the length of the *arc s* of a curve measured from a given fixed point on the curve and the *angle* $\psi$ between the tangents at its extremities is called the *instrinsic equation of the curve*.

The expression $\dfrac{ds}{d\psi}$ for radius of curvature is suitable only for those curves whose intrinsic equations are given. We now proceed to find formulae for radius of curvature when equations of curves are given in other forms.

In this chapter, we shall need few results of chapter on "derivatives of arcs."

All these results can be obtained from the following figures :

NOTES

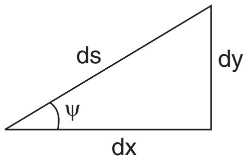



Fig. (*i*)

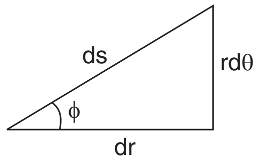



Fig. (*ii*)

$$\therefore \qquad \cos\psi = \frac{dx}{ds},\ \sin\psi = \frac{dy}{ds},\ \tan\psi = \frac{dy}{dx}$$

Again, $\quad \cos\phi = \dfrac{dr}{ds},\ \sin\phi = r\dfrac{d\theta}{ds},\ \tan\phi = r\dfrac{d\theta}{dr}.$

## RADIUS OF CURVATURE FOR CARTESIAN EQUATION

(*i*) *When the equation of the curve is given in the explicit form y = f(x).*

We know that slope of the tangent at any point = $\tan\psi = \dfrac{dy}{dx}$

Differentiating both sides w.r.t. *s*,

$$\sec^2\psi\frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

or
$$\sec^2\psi\cdot\frac{1}{\rho} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$$

or
$$\sec^2\psi\cdot\frac{1}{\rho} = \frac{d^2y}{dx^2}\cos\psi$$

$$\left[\because \text{ From Fig. } (i) \text{ of Art. 4, } \cos\psi = \frac{dx}{ds}\right]$$

$$\therefore \qquad \rho = \frac{\sec^2\psi}{\cos\psi\dfrac{d^2y}{dx^2}} = \frac{\sec^3\psi}{\dfrac{d^2y}{dx^2}} = \frac{(1+\tan^2\psi)^{3/2}}{\dfrac{d^2y}{dx^2}}$$

*i.e.*,
$$\rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}} = \frac{(1+y_1^2)^{3/2}}{y_2} \qquad \ldots(1)$$

## CONVENTION OF SIGNS

The positive root is taken in numerator of (1), therefore radius of curvature, ρ, will be positive when $y_2$ is positive (*i.e.,* when curve is concave upwards) and *negative,* when $y_2$ is negative (*i.e.,* when curve is concave downwards). *In practice numerical value of* ρ *is taken.* Since at a point of inflexion $y_2$ is zero therefore curvature of a curve at a point of inflexion is zero.

**Cor.** When equation of curve is given in the form $x = f(y)$, then by interchanging $x$ and $y$, we get

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

[**Hint for proof.** $\because$ $\tan\psi = \frac{dy}{dx}$

$\therefore$ $\cot\psi = \frac{dx}{dy}$. Differentiate w.r.t. $s$.]

(*ii*) *When the equation of curve is given in* **parametric form.**

Let $x = f(t)$, $y = \phi(t)$, be the parametric equations of the curve, $t$ being the parameter.

Then $$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{y'}{x'}.$$

where dashes denote differentiations w.r.t. $t$.

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right).\frac{dt}{dx}$$

$$= \frac{x'y'' - y'.x''}{(x')^2}.\frac{1}{x'} = \frac{x'.y'' - y'x''}{(x')^3}$$

$\therefore$ Substituting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in formula (1) and simplifying, we get

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}.$$

(*iv*) *When x and y are given as function of the length of arc s, measured from a fixed point on the curve.*

Let the curve be given by the equations $x = f(s)$ and $y = \phi(s)$, where $s$ denotes the length of arc measured from a *fixed point* on the curve.

We know [from Fig. (*i*) of Art. 4] that

$$\cos\psi = \frac{dx}{ds} \quad \ldots(i), \qquad \text{and} \qquad \sin\psi = \frac{dy}{ds} \quad \ldots(ii)$$

Differentiating (*i*) w.r.t. $s$, we have $-\sin\psi.\frac{d\psi}{ds} = \frac{d^2x}{ds^2}$ ...(*iii*)

or $$-\frac{dy}{ds}.\frac{1}{\rho} = \frac{d^2x}{ds^2} \qquad [\because \text{ of } (ii)]$$

$\therefore$ $$\rho = -\frac{dy}{ds}\Big/\frac{d^2x}{ds^2}.$$

Again differentiating (*ii*) w.r.t. $s$, we get $\cos\psi\frac{d\psi}{ds} = \frac{d^2y}{ds^2}$ ...(*iv*)

or $$\frac{dx}{ds}.\frac{1}{\rho} = \frac{d^2y}{ds^2} \qquad [\because \text{ of } (i)]$$

NOTES

$\therefore \qquad \rho = \dfrac{\mathbf{dx}}{\mathbf{ds}} \Big/ \dfrac{\mathbf{d^2y}}{\mathbf{ds^2}}.$

Squaring and adding (*iii*) and (*iv*), we get

$$\left(\frac{d\psi}{ds}\right)^2 [\sin^2\psi + \cos^2\psi] = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 \quad \text{or} \quad \frac{1}{\rho^2} = \left(\frac{\mathbf{d^2x}}{\mathbf{ds^2}}\right)^2 + \left(\frac{\mathbf{d^2y}}{\mathbf{ds^2}}\right)^2.$$

## SOLVED EXAMPLES

**Example 1.** *Prove that the radius of curvature for the catenary y = c cosh x/c is equal to the portion of the normal intercepted between the curve and the x-axis and that it varies as the square of the ordinate.*

**Sol.** Equation of the curve is $y = c \cosh \dfrac{x}{c}$ ...(1)

Diff. (1) w.r.t. $x$, $\quad y_1 = c \sinh \dfrac{x}{c} \times \dfrac{1}{c}$ $\quad \Big| \because \dfrac{d}{d\theta}(\cosh\theta) = \sinh\theta$

or $\quad y_1 = \sinh \dfrac{x}{c}$

Again diff. w.r.t. $x$, $y_2 = \cosh \dfrac{x}{c} \times \dfrac{1}{c} = \dfrac{1}{c}\cosh\dfrac{x}{c}$

$$\therefore \qquad \rho = \frac{(1+{y_1}^2)^{3/2}}{y_2} = \frac{\left(1+\sinh^2\dfrac{x}{c}\right)^{3/2}}{\dfrac{1}{c}\cosh\dfrac{x}{c}}$$

$$= \frac{\left(\cosh^2\dfrac{x}{c}\right)^{3/2}}{\dfrac{1}{c}\cosh\dfrac{x}{c}} \qquad \Bigg| \begin{array}{l} \because \cosh^2\theta - \sinh^2\theta = 1 \\ \therefore \cosh^2\theta = 1+\sinh^2\theta \end{array}$$

or $\quad \rho = \dfrac{c\cosh^3 x/c}{\cosh x/c} = c\cosh^2\dfrac{x}{c}$ ...(2)

Now portion of the normal intercepted between the curve and the $x$-axis.

$= \text{Length of normal} = \mathbf{y}\sqrt{\mathbf{1}+\mathbf{y_1}^2}$

$= c\cosh\dfrac{x}{c}\sqrt{1+\sinh^2\dfrac{x}{c}} = c\cosh\dfrac{x}{c}.\cosh\dfrac{x}{c} = c\cosh^2\dfrac{x}{c}$ ...(3)

From (2) and (3), we have $\rho$ = Length of Normal

Again $\dfrac{\rho}{(\text{Ordinate})^2} = \dfrac{\rho}{y^2} = \dfrac{c\cosh^2 x/c}{c^2\cosh^2 x/c} = \dfrac{1}{c}$ = constant.

$\therefore \quad \rho$ varies as $y^2$ *i.e.,* as square of the ordinate.

**Remark.** If the equation of the curve is given to be

$y = \dfrac{c}{2}(e^{x/c} + e^{-x/c})$, we should write it as

$y = c\left(\dfrac{e^{x/c}+e^{-x/c}}{2}\right) = c\cosh\dfrac{x}{c}$ $\quad \Big| \because \cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$

NOTES

**Example 2.** *In the cycloid* $x = a(\theta + \sin\theta)$ ; $y = a(1 - \cos\theta)$, *prove that*

$$\rho = 4a \cos \tfrac{1}{2}\theta.$$

**Sol.** Here $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

$\therefore$ $x' = a(1 + \cos\theta)$, $y' = a\sin\theta$

and $x'' = -a\sin\theta$, $y'' = a\cos\theta$

$$\therefore \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{a^3\left\{(1+\cos\theta)^2 + \sin^2\theta\right\}^{3/2}}{a^2(1+\cos\theta)\cos\theta + a^2\sin\theta\,.\,\sin\theta}$$

$$= \frac{a^3(1+\cos^2\theta + 2\cos\theta + \sin^2\theta)^{3/2}}{a^2(\cos\theta + \cos^2\theta + \sin^2\theta)} = \frac{a(1+1+2\cos\theta)^{3/2}}{(\cos\theta + 1)}$$

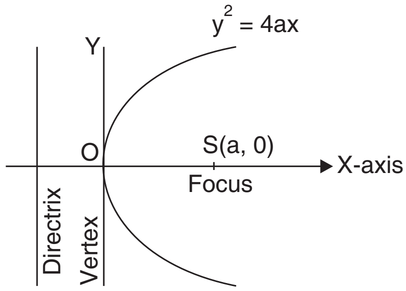
$$= \frac{a(2+2\cos\theta)^{3/2}}{(1+\cos\theta)} = a\,.\,2^{3/2}\frac{(1+\cos\theta)^{3/2}}{(1+\cos\theta)} = a\,.\,2^{3/2}(1+\cos\theta)^{1/2}$$

$$= a\,.\,2^{3/2}\,.\,(2\cos^2\tfrac{1}{2}\theta)^{1/2} = 4a\cos\tfrac{1}{2}\theta.$$

**Remark.** We can also use the formula $\rho = \dfrac{(1+y_1{}^2)^{3/2}}{y_2}$ for parametric equations.

## NOTE ON PARABOLA $y^2 = 4ax$

The shape of the parabola $y^2 = 4ax$ is as shown in the adjoining figure.




O(0, 0) is the VERTEX of this parabola.

$x$-axis is the AXIS of this parabola.

$y$-axis is the **tangent at the vertex** to this parabola.

For the parabola $y^2 = 4ax$ ; **4a,** the coefficient of $x$ in this equation is called length of **Latus Rectum** of the parabola.

The point (**a, 0**) on the axis of parabola is called **Focus** of the parabola.

The line $x = -a$ is DIRECTRIX of the parabola.

Any chord of the parabola passing through the focus is called **a focal chord.**

Any point $(x, y)$ on the parabola $y^2 = 4ax$ is ($\mathbf{at^2, 2at}$).

[$\because$ $(at^2, 2at)$ satisfies the equation of parabola]

This point $(at^2, 2at)$ is briefly written as point $t$ on the parabola. $t$ is also called "parameter".

Also $\mathbf{x = at^2, y = 2at}$ are called **Parametric Equations** of the parabola $y^2 = 4ax$.

### SOLVED EXAMPLES

**Example 3.** *Show that for the parabola* $y^2 = 4ax$, $\rho^2$ *varies as* $(SP)^3$ *where* $\rho$ *is the radius of curvature at any point P of the parabola and S is the focus of the parabola.*

**Sol.** Equation of the parabola is $y^2 = 4ax$.

Any point P$(x, y)$ on the parabola is $(at^2, 2at)$

*i.e.,* parametric equations of the parabola are

$$x = at^2, \qquad y = 2at$$

$$\therefore \quad x' = 2at, \qquad y' = 2a$$

$$x'' = 2a, \qquad y'' = 0$$

$$\therefore \quad \rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(4a^2t^2 + 4a^2)^{3/2}}{0 - 4a^2}$$

or $$\rho = -\frac{(4a^2)^{3/2}(t^2+1)^{3/2}}{4a^2} = -(4a^2)^{1/2}(1+t^2)^{3/2}$$

or $$\rho = 2a(1+t^2)^{3/2} \text{ (Numerically)}$$

Also SP (where S$(a, 0)$ is focus and P is $(at^2, 2at)$

$$= \sqrt{(at^2 - a)^2 + (2at - 0)^2} \quad \Big| \quad \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{a^2t^4 + a^2 - 2a^2t^2 + 4a^2t^2}$$

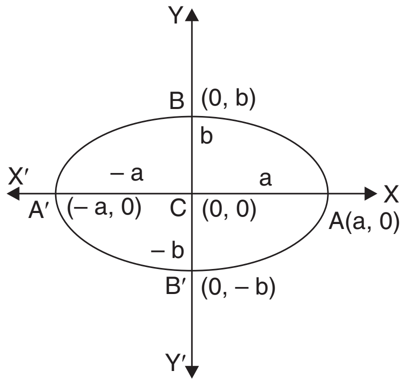
$$= \sqrt{a^2t^4 + a^2 + 2a^2t^2} = \sqrt{(at^2 + a)^2} = at^2 + a = a(1+t^2)$$

$$\therefore \quad \frac{\rho^2}{(\text{SP})^3} = \frac{[2a(1+t^2)^{3/2}]^2}{[a(1+t^2)]^3} = \frac{4a^2(1+t^2)^3}{a^3(1+t^2)^3} = \frac{4}{a} = \text{Constant.}$$

$\therefore$ $\rho^2$ varies as $(\text{SP})^3$.

## NOTE ON ELLIPSE $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is as shown in the adjoining figure.




C(0, 0) is called CENTRE of the ellipse.

$x$-axis is called MAJOR AXIS of the ellipse and length of major axis is $2a$.

$y$-axis is called MINOR AXIS of the ellipse and length of minor-axis is $2b$.

**Eccentricity** $e$ of the ellipse is given by

$$\mathbf{b^2 = a^2(1 - e^2).}$$

Any point $(x, y)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is **(a cos θ, b sin θ)**

[ $\because$ $(a \cos \theta, b \sin \theta)$ satisfies the equation of the ellipse]

The point $(a \cos \theta, b \sin \theta)$ is briefly written as point θ on the ellipse.

θ is called **Parameter** or **Eccentric Angle** of the point.

### Conjugate Diameters of an Ellipse

Two diameters of an ellipse are said to be **conjugate** if each bisects chords parallel to the other.

If CP and CD are two semi-conjugate diameters of an ellipse and P is point θ *i.e.,* P is $(a\cos\theta, b\sin\theta)$ ; then the point D is $\frac{\pi}{2}+\theta$ *i.e.,* D is $\left[a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right]$.

## SOLVED EXAMPLES

**Example 4.** *If CP, CD are a pair of semi-conjugate diameters of an ellipse of semi-axes of lengths a and b, prove that the radius of curvature at P =* $\frac{CD^3}{ab}$ *where C is the centre of the ellipse.*

**Sol.** Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Any point P(*x*, *y*) on the ellipse is $(a\cos\theta, b\sin\theta)$ *i.e.,* parametric equations of the ellipse are

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$\therefore \qquad x' = -a\sin\theta \qquad y' = b\cos\theta$$

$$x'' = -a\cos\theta \qquad y'' = -b\sin\theta$$

$$\therefore \qquad \rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''} = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta+ab\cos^2\theta}$$

or
$$\rho = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab} \qquad \ldots(1)$$

∵ CP and CD are semi-conjugate diameters of the ellipse and point P is $(a\cos\theta, b\sin\theta)$, therefore

$$\text{D is } \left[a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right]$$

or
$$\text{D is } (-a\sin\theta, b\cos\theta)$$

∴ CD where C(0, 0) is centre of the ellipse

$$= \sqrt{(-a\sin\theta-0)^2+(b\cos\theta-0)^2} = \sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

Putting this value of $\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ = CD in (1), we have $\rho = \frac{\text{CD}^3}{ab}$.

**Example 5.** *Show that the radius of curvature at any point of the curve* $x^{2/3}+y^{2/3}=a^{2/3}$ *is equal to three times the length of the perpendicular from the origin on the tangent.*

**Sol.** Equation of the curve is $x^{2/3}+y^{2/3}=a^{2/3}$ $\qquad \ldots(1)$

Dividing by $a^{2/3}$ ; $\qquad \left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3}=1$

or
$$\left(\left(\frac{x}{a}\right)^{1/3}\right)^2+\left(\left(\frac{y}{a}\right)^{1/3}\right)^2=1$$

Comparing with $\cos^2\theta+\sin^2\theta=1$, we have

$$\left(\frac{x}{a}\right)^{1/3}=\cos\theta \quad \text{and} \quad \left(\frac{y}{a}\right)^{1/3}=\sin\theta$$

NOTES

Cubing both equations $\frac{x}{a} = \cos^3 \theta$ and $\frac{y}{a} = \sin^3 \theta$

or $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$ ...(2)

Equations (2) are parametric equations of curve (1).

(These equations (2) of curve (1) should be preferred in this chapter)

**Let us find ρ for curve (1) using Equations (2)**

Differentiating eqns. (2) w.r.t. θ,

$$\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta \text{ and } \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\therefore \quad y_1 = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$$

Again differentiating w.r.t. $x$,

$$y_2 = \frac{d^2y}{dx^2} = -\sec^2 \theta \frac{d\theta}{dx}$$

$$= \frac{-1}{\cos^2 \theta} \cdot \frac{1}{-3a \cos^2 \theta \sin \theta} = \frac{1}{3a \cos^4 \theta \sin \theta}$$

We know that, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Putting values of $y_1$ and $y_2$,

$$\rho = \frac{(1+\tan^2 \theta)^{3/2}}{\left(\frac{1}{3a \cos^4 \theta \sin \theta}\right)} = 3a \cos^4 \theta \sin \theta \left(\sec^2 \theta\right)^{3/2}$$

$$= 3a \cos^4 \theta \sin \theta \sec^3 \theta = 3a \cos^4 \theta \sin \theta \frac{1}{\cos^3 \theta}$$

or $\rho = 3a \cos \theta \sin \theta$ ...(3)

Now let us find the equation of the tangent at any point $(x, y)$ *i.e.,* $(a \cos^3 \theta, a \sin^3 \theta)$ (using equations (2) to curve (1)).

We know that slope of the tangent at any point

$$(x, y) = \frac{dy}{dx} = -\tan \theta = -\frac{\sin \theta}{\cos \theta}$$

∴ Equation of the tangent at any point $(a \cos^3 \theta, a \sin^3 \theta)$

is $y - a \sin^3 \theta = -\frac{\sin \theta}{\cos \theta}(x - a \cos^3 \theta)$ | $y - y_1 = m(x - x_1)$

Cross-multiplying $y \cos \theta - a \sin^3 \theta \cos \theta = -x \sin \theta + a \cos^3 \theta \sin \theta$

or $x \sin \theta + y \cos \theta - a \sin^3 \theta \cos \theta - a \cos^3 \theta \sin \theta = 0$

or $x \sin \theta + y \cos \theta - a \sin \theta \cos \theta (\sin^2 \theta + \cos^2 \theta) = 0$

or $x \sin \theta + y \cos \theta - a \sin \theta \cos \theta = 0$ ...(4)

$p$ = length of ⊥ from origin (0, 0) on tangent (4)

$$= \frac{|0 + 0 - a \sin \theta \cos \theta|}{\sqrt{\sin^2 \theta + \cos^2 \theta}}$$ $\left| \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \right.$

$= a \sin \theta \cos \theta$ ...(5)

From (3) and (5), we have $\rho = 3p$

*i.e.,* radius of curvature at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is equal to three times the length of the perpendicular from the origin on the tangent.

NOTES

## SOLVED EXAMPLES

**Example 6.** *Show that the radius of curvature of the curve* $\sqrt{x} + \sqrt{y} = \sqrt{a}$ *is minimum at the point* $\left(\frac{1}{4}a, \frac{1}{4}a\right)$.

**Sol.** Equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{a}$ ...(1)

Diff. both sides of Eqn. (1) w.r.t. $x$,

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{1}{2}\frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\frac{\sqrt{y}}{\sqrt{x}} \quad \text{...(2)}$$

Again diff. (2) w.r.t. $x$,

$$\frac{d^2y}{dx^2} = -\frac{\left[\sqrt{x}\,\frac{1}{2}y^{-1/2}\frac{dy}{dx} - \sqrt{y}\,.\,\frac{1}{2}x^{-1/2}\right]}{x}$$

$$= -\frac{1}{2}\frac{\left[\frac{\sqrt{x}}{\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{\sqrt{x}}\right]}{x} \quad \text{[By (2)]}$$

$$= -\frac{1}{2}\frac{\left(-1 - \frac{\sqrt{y}}{\sqrt{x}}\right)}{x} = \frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{x\sqrt{x}}\right) = \frac{1}{2}\cdot\frac{\sqrt{a}}{x\sqrt{x}} \quad \text{[By (1)] ...(3)}$$

Putting these values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in $\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$, we have

$$\rho = \frac{\left(1+\frac{y}{x}\right)^{3/2}}{\left(\frac{\sqrt{a}}{2x\sqrt{x}}\right)} = \frac{(x+y)^{3/2}}{x^{3/2}}\,\frac{2x\sqrt{x}}{\sqrt{a}} \quad \text{or} \quad \rho = \frac{2}{\sqrt{a}}(x+y)^{3/2} \quad \text{...(4)}$$

**To find minimum value of** $\rho$

Diff. (4) w.r.t. $x$, $\quad \frac{d\rho}{dx} = \frac{2}{\sqrt{a}}\,\frac{3}{2}(x+y)^{1/2}\left(1+\frac{dy}{dx}\right)$

or

$$\frac{d\rho}{dx} = \frac{3}{\sqrt{a}}(x+y)^{1/2}\left(1+\frac{dy}{dx}\right) \quad \text{...(5)}$$

Again diff. (5) w.r.t. $x$,

$$\frac{d^2\rho}{dx^2} = \frac{3}{\sqrt{a}}\left[(x+y)^{3/2}\frac{d^2y}{dx^2} + \frac{1}{2}(x+y)^{-1/2}\left(1+\frac{dy}{dx}\right)\left(1+\frac{dy}{dx}\right)\right] \quad \text{...(6)}$$

NOTES

**Put** $\frac{\mathbf{d\rho}}{\mathbf{dx}} = \mathbf{0}$ $\therefore$ $\frac{3}{\sqrt{a}}(x+y)^{1/2}\left(1+\frac{dy}{dx}\right) = 0$

But $\frac{3}{\sqrt{a}} \neq 0$ also $x + y \neq 0$ [By (1)]

$\therefore$ $1+\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -1$

$\therefore$ From (2), $-\frac{\sqrt{y}}{\sqrt{x}} = -1$ or $\sqrt{y} = \sqrt{x}$ $\therefore$ $y = x$ ...(7)

Let us solve Eqns. (1) and (7) for $x$ and $y$.

Putting $y = x$ from (7) in (1), $\sqrt{x} + \sqrt{x} = \sqrt{a}$ or $2\sqrt{x} = \sqrt{a}$

$\therefore$ $\sqrt{x} = \frac{\sqrt{a}}{2}$ $\therefore$ $x = \frac{a}{4}$

$\therefore$ From (7) $y = x = \frac{a}{4}$

$\therefore$ Turning point for ρ to be minimum is $\left(\frac{a}{4}, \frac{a}{4}\right)$

At $x = y = \frac{a}{4}$, from (3), $\frac{d^2y}{dx^2} = \frac{1}{2}\frac{\sqrt{a}}{\frac{a}{4}\cdot\sqrt{\frac{a}{4}}} = \frac{1}{2}\cdot\frac{4}{a}\cdot\sqrt{\frac{4}{a}}\sqrt{a} = \frac{4}{a}$

Putting $x = \frac{a}{4}, y = \frac{a}{4}, \frac{dy}{dx} = -1, \frac{d^2y}{dx^2} = \frac{4}{a}$ in (6),

$$\frac{d^2\rho}{dx^2} = \frac{3}{\sqrt{a}}\left[\left(\frac{a}{4}+\frac{a}{4}\right)^{3/2}\cdot\frac{4}{a}\right] > 0$$

$\therefore$ ρ is minimum at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$

## EXERCISE A

**1.** Find the radius of curvature at the point $(s, \psi)$ on the following curves :

(*i*) $s = a \log \tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)$ (*ii*) $s = 4a \sin \psi$ [cycloid].

**2.** Find the radius of curvature at the given point of the following curves :

(*i*) Rectangular hyperbola $xy = c^2$ at the point $(x, y)$.

[**Hint.** From the equation of the curve $y = \frac{c^2}{x}$.]

(*ii*) $y = 4 \sin x - \sin 2x$ at $x = \frac{\pi}{2}$. (*iii*) $\sqrt{x} + \sqrt{y} = 1$ at the point $(\frac{1}{4}, \frac{1}{4})$.

**3.** Find the radius of curvature at any point of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

**4.** Find the radius of curvature of the curve $y = e^x$, at the point where it crosses the $y$-axis.

**5.** Find the radius of curvature at the origin of the two branches of the curve given by

$$x = 1 - t^2, y = t - t^3.$$

[**Hint.** At the origin $x = 0, y = 0$

$\therefore$ $1 - t^2 = 0$ and $t - t^3 = 0$. Common values of $t$ on solving are ± 1.]

**6.** Prove that the radius of curvature of the curve $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ is $\frac{y^2}{a}$.

[**Hint.** We know that $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$,

$\therefore$ Equation of the curve becomes $y = a\cosh\frac{x}{a}$. Now see example 1.]

**7.** For the curve $y = a\,e^{x/a}$, prove that $\rho = a\sec^2\theta\,\mathrm{cosec}\,\theta$, where $\theta = \tan^{-1}\frac{y}{a}$.

**8.** For the curve $y = \frac{ax}{a+x}$, prove that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$.

**9.** Find the radius of curvature for the curve $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$ at the points where it touches the co-ordinate axes.

$\Bigg[$**Hint.** The point where the curve touches $x$-axis ; $\frac{dy}{dx} = 0$. The point where the curve touches $y$-axis, $\frac{dx}{dy} = 0.\Bigg]$

**10.** Show that for the curve $x = a\cos\theta\,(1 + \sin\theta)$ and $y = a\sin\theta\,(1 + \cos\theta)$, the radius of curvature is $a$, at the point for which the value of the parameter $\theta$ is $-\frac{\pi}{4}$.

**11.** Prove that for the curve $x^3 + y^3 = 3axy$, the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is numerically equal to $\frac{3a\sqrt{2}}{16}$.

**12.** Show that the radius of curvature at the point $(-2a, 2a)$ on the curve $x^2y = a(x^2 + y^2)$ is $2a$.

**13.** Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$.

**14.** If $\rho_1$ and $\rho_2$ be the radii of curvature at the extremities of a focal chord of a parabola $y^2 = 4ax$, prove that $(\rho_1)^{-2/3} + (\rho_2)^{-2/3} = (2a)^{-2/3}$

[**Hint.** If $t_1$ and $t_2$ are the parameters of the two extremities of a focal chord of a parabola, then $t_1t_2 = -1$.]

### Answers

**1.** (*i*) $a\sec\psi$ (*ii*) $4a\cos\psi$

**2.** (*i*) $\frac{(x^4 + c^4)^{3/2}}{2c^2x^3}$ (*ii*) $\frac{5\sqrt{5}}{4}$ (*iii*) $\frac{1}{\sqrt{2}}$

**3.** $a\theta$ **4.** $\sqrt{8}$ **5.** $2\sqrt{2}$ **9.** $\frac{2a^2}{b}, \frac{2b^2}{a}$

**13.** $(\frac{9}{8}, 3)$; $(\frac{9}{8}, -3)$

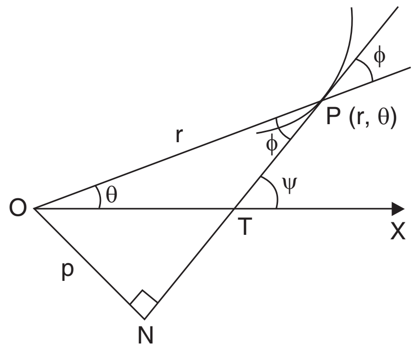
## RADIUS OF CURVATURE FOR POLAR EQUATIONS

*To find the radius of curvature for the curve $r = f(\theta)$ or $f(r, \theta) = 0$*

Let the tangent at any point $P(r, \theta)$ to curve make an angle $\psi$ with the initial line OX.

NOTES

Then from the figure, we have




$$\psi = \theta + \phi$$

$$\therefore \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

or $$\frac{1}{\rho} = \frac{d\theta}{ds}\left[1 + \frac{d\phi}{d\theta}\right]$$

$$= \frac{\sin\phi}{r}\left[1 + \frac{d\phi}{d\theta}\right] \quad \ldots(1)$$

$$\left[\because \text{ From Fig. } (ii) \text{ of Art. 4, } \sin\phi = r\frac{d\theta}{ds} \quad \therefore \quad \frac{d\theta}{ds} = \frac{\sin\phi}{r}\right]$$

Again from Fig. (*ii*) Art. 4, $\tan\phi = r \,.\, \dfrac{d\theta}{dr} = \dfrac{r}{\dfrac{dr}{d\theta}} = \dfrac{r}{r_1}$ ...(2)

*To find* $\dfrac{d\phi}{d\theta}$

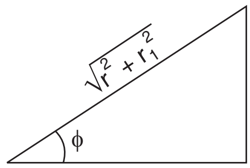
Differentiating (2) w.r.t. θ,

$$\sec^2\theta \, \frac{d\phi}{d\theta} = \frac{r_1 \,.\, r_1 - rr_2}{r_1^2} = \frac{r_1^2 - rr_2}{r_1^2}$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2 \sec^2\phi} = \frac{r_1^2 - rr_2}{r_1^2\left[1 + \dfrac{r^2}{r_1^2}\right]} = \frac{r_1^2 - rr_2}{r_1^2 + r^2} \qquad \left|\because \tan\phi = \frac{r}{r_1}\right.$$

Also $\quad \tan\phi = \dfrac{r}{r_1}$ gives $\sin\phi = \dfrac{r}{\sqrt{r^2 + r_1^2}}$

Substituting these values of $\dfrac{d\phi}{d\theta}$ and $\sin\phi$ in (1), we get




$$\frac{1}{\rho} = \frac{1}{\sqrt{r^2 + r_1^2}}\left[1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2}\right]$$

or $$\frac{1}{\rho} = \frac{r^2 + 2r_1^2 - rr_2}{(r^2 + r_1^2)^{3/2}} \quad \therefore \quad \rho = \frac{\mathbf{(r^2 + r_1^2)^{3/2}}}{\mathbf{r^2 + 2r_1^2 - rr_2}} \quad \ldots(3)$$

**Cor.** *If equation of a curve is given in the form* $u = f(\theta)$*, where*

$$u = \frac{1}{r} \qquad i.e., \qquad r = \frac{1}{u}, \text{ then we have } \; r_1 = -\frac{1}{u^2} \cdot \frac{du}{d\theta}$$

and $$r_2 = -\frac{1}{u^2} \cdot \frac{d^2u}{d\theta^2} + \frac{2}{u^3} \cdot \left(\frac{du}{d\theta}\right)^2$$

Substituting these values of $r_1$ and $r_2$ in (3), we get

$$\rho = \frac{\left[\dfrac{1}{u^2} + \dfrac{1}{u^4}\left(\dfrac{du}{d\theta}\right)^2\right]^{3/2}}{\dfrac{1}{u^2} + 2\,\dfrac{1}{u^4}\left(\dfrac{du}{d\theta}\right)^2 - \dfrac{1}{u}\left[-\dfrac{1}{u^2} \cdot \dfrac{d^2u}{d\theta^2} + \dfrac{2}{u^2}\left(\dfrac{du}{d\theta}\right)^2\right]}$$

or
$$\rho = \frac{\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]^{3/2}}{u^3\left[u + \frac{d^2u}{d\theta^2}\right]} - \frac{\mathbf{(u^2 + u_1{}^2)^{3/2}}}{\mathbf{u^3(u + u_2)}}$$

where $u_1 = \frac{du}{d\theta}$ and $u_2 = \frac{d^2u}{d\theta^2}$.

7. Radius of Curvature for Pedal Equations $\left(\textbf{To prove } \rho = \mathbf{r}\,\frac{\mathbf{dr}}{\mathbf{dp}}\right)$

Let the pedal equation of the curve be $p = f(r)$.

From the figure (*refer to figure of article 9.6*), we have

$$\psi = \theta + \phi \quad \therefore \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \quad i.e., \quad \frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{ds} \qquad \ldots(i)$$

But we know that $p = r \sin \phi$

Differentiating this w.r.t. $r$, we have

$$\frac{dp}{dr} = \sin\phi \,.\, 1 + r\cos\phi \,.\, \frac{d\phi}{dr} = r \,.\, \frac{d\theta}{ds} + r \,.\, \frac{dr}{ds}\frac{d\phi}{dr}$$

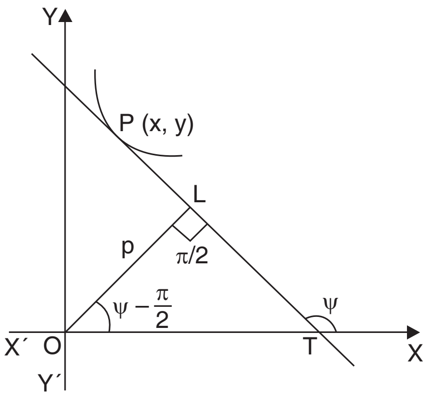
$$\left[\because \ \sin\phi = r\,.\frac{d\theta}{ds}, \cos\phi = \frac{dr}{ds}\right]$$

$$= r\left[\frac{d\theta}{ds} + \frac{d\phi}{ds}\right] = r\,.\,\frac{1}{\rho} \qquad [\because \text{ of relation } (i)]$$

or
$$\frac{dp}{dr} = r\,.\frac{1}{\rho} \qquad \therefore \qquad \rho = \frac{r}{dp/dr} = \mathbf{r}\,.\,\frac{\mathbf{dr}}{\mathbf{dp}}.$$

## RADIUS OF CURVATURE FOR TANGENTIAL POLAR EQUATIONS p = f(ψ) $\left(\textbf{To prove } \rho = \mathbf{p} + \frac{\mathbf{d^2p}}{\mathbf{d\psi^2}}\right)$

A *relation between* perpendicular $p$ from the origin on any tangent to a curve and *angle* ψ which this tangent makes with $x$-axis, is called the **tangential polar equation** of the curve.




Let $p$ be length of the perpendicular OL drawn from the origin on the tangent to curve at the point P($x$, $y$), then OL makes an angle $\psi - \frac{\pi}{2}$ with the positive direction of $x$-axis.

∴ Equation of the tangent PT is

$$p = X\cos\left(\psi - \frac{\pi}{2}\right) + Y\sin\left(\psi - \frac{\pi}{2}\right)$$

[Normal form $x\cos\alpha + y\sin\alpha = p$]

or $\quad p = X\sin\psi - Y\cos\psi$

where X, Y are Cartesian co-ordinates of any point on this tangent.

NOTES

As point P($x$, $y$) lies on this tangent

$\therefore \quad p = x \sin \psi - y \cos \psi \quad \ldots(i)$

Differentiating both sides of ($i$) w.r.t. $\psi$, we get

$$\frac{dp}{d\psi} = x \cos \psi + \sin \psi \cdot \frac{dx}{d\psi} + y \sin \psi - \cos \psi \frac{dy}{d\psi}$$

$$= x \cos \psi + y \sin \psi + \sin \psi \cdot \frac{dx}{ds} \cdot \frac{ds}{d\psi} - \cos \psi \cdot \frac{dy}{ds} \cdot \frac{ds}{d\psi}$$

$$= x \cos \psi + y \sin \psi + \sin \psi \, \rho \cdot \cos \psi - \cos \psi \cdot \rho \sin \psi$$

$$\left[\because \frac{dx}{ds} = \cos \psi, \frac{dy}{ds} = \sin \psi\right]$$

$$= x \cos \psi + y \sin \psi$$

Differentiating again w.r.t. $\psi$, we get

$$\frac{d^2 p}{d\psi^2} = -x \sin \psi + \cos \psi \cdot \frac{dx}{d\psi} + y \cos \psi + \sin \psi \cdot \frac{dy}{d\psi}$$

$$= -x \sin \psi + y \cos \psi + \cos \psi \cdot \frac{dx}{ds} \cdot \frac{ds}{d\psi} + \sin \psi \cdot \frac{dy}{ds} \cdot \frac{ds}{d\psi}$$

$$= (-x \sin \psi + y \cos \psi) + \cos \psi \cdot \cos \psi \cdot \rho + \sin \psi \cdot \sin \psi \cdot \rho$$

$$= -p + \rho[\cos^2 \psi + \sin^2 \psi] \quad [\because \text{ of relation } (i)]$$

Hence $\quad \boldsymbol{\rho = p + \dfrac{d^2p}{d\psi^2}}$.

## SOLVED EXAMPLES

**Example 7.** *Find the radius of curvature for curve $r^n = a^n \cos n\theta$.*

**Sol. First Method.** Taking logarithms of both sides, we have

$$n \log r = n \log a + \log \cos n\theta$$

Now differentiating, $\quad \dfrac{n}{r} \cdot \dfrac{dr}{d\theta} = 0 + \dfrac{1}{\cos n\theta}(-n \sin n\theta)$

$$= -n \tan n\theta \quad \ldots(i)$$

$\therefore \quad r_1 = -r \tan n\theta$

Differentiating again,

$$r_2 = -rn \cdot \sec^2 n\theta - r_1 \cdot \tan n\theta$$

$$= -rn \sec^2 n\theta + r \tan^2 n\theta \quad \ldots(ii)$$

$$\therefore \quad \rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta + r^2 n \sec^2 n\theta - r^2 \tan^2 n\theta} \quad [\because \text{ of } (i) \text{ and } (ii)]$$

$$= \frac{r^3 \sec^3 n\theta}{(n+1) r^2 \sec^2 n\theta} = \frac{r \cdot \sec n\theta}{n+1} = \frac{r}{n+1} \cdot \frac{1}{\cos n\theta}$$

$$= \frac{r}{(n+1)\dfrac{r^n}{a^n}} = \frac{a^n}{(n+1) r^{n-1}} \quad \left[\because \frac{r^n}{a^n} = \cos n\theta, \text{ from eq. of curve}\right]$$

**Alternative Method.** We first change of polar equation of the curve into the pedal one and find the value of $r \cdot \frac{dr}{dp}$.

As found in (*i*) above, we have $\frac{dr}{d\theta} = -r \tan n\theta$

$$\therefore \qquad \tan \phi = r\frac{d\theta}{dr} = \frac{r}{\frac{dr}{d\theta}} = -\cot n\theta = \tan\left(\frac{\pi}{2} + n\theta\right)$$

*i.e.,*
$$\phi = \frac{1}{2}\pi + n\theta \qquad \ldots(iii)$$

Now
$$p = r \sin \phi = r \sin\left(\frac{1}{2}\pi + n\theta\right) \qquad [\because \text{ of } (iii)]$$

or
$$p = r \cos n\theta = r \cdot \frac{r^n}{a^n} \qquad \text{[From the equation of the curve]}$$

Hence pedal equation is $p = \dfrac{r^{n+1}}{a^n}$

Differentiating, $\dfrac{dp}{dr} = \dfrac{(n+1)r^n}{a^n}$

$$\therefore \qquad \rho = r \cdot \frac{dr}{dp} = r \cdot \frac{a^n}{(n+1)\,r^n} = \frac{a^n}{(n+1)\,r^{n-1}}.$$

**Note.** The above question may also be put in the following form : *Show that for the curve $a^n p = r^{n+1}$; ρ varies inversely as the (n – 1)th power of radius vector.*

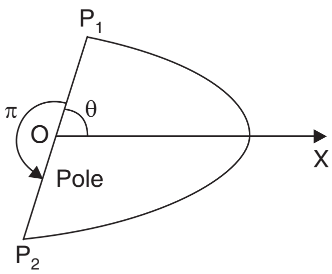
**Remark. To transform polar equation to pedal equation**

1. Find $\phi$ from the formula $\tan \phi = \dfrac{r}{\left(\dfrac{dr}{d\theta}\right)}$.
2. Put this value of $\phi$ in $p = r \sin \phi$.
3. Eliminate θ.

**Example 8.** *If $\rho_1$, $\rho_2$ be the radii of curvature at the extremities of any chord through the pole of the cardioide $r = a(1 + \cos \theta)$ ; show that $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$.*

**Sol.** Let $\rho_1$ and $\rho_2$ be the radii of curvature at the extremities $P_1$ and $P_2$ of the chord $P_1OP_2$ of the curve

$$r = a(1 + \cos \theta) \qquad \ldots(1)$$




Diff. (1) w.r.t. θ, $r_1 = a(0 - \sin \theta) = -a \sin \theta$

Again diff. w.r.t. θ, $r_2 = -a \cos \theta$

We know that
$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{[a^2(1+\cos\theta)^2 + a^2 \sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2\cos\theta\,(1+\cos\theta)}$$

$$= \frac{a^3\,[1+\cos^2\theta + 2\cos\theta + \sin^2\theta]^{3/2}}{a^2\,[1+\cos^2\theta + 2\cos\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta]}$$

NOTES

$$= \frac{a[2 + 2\cos\theta]^{3/2}}{[3 + 3\cos\theta]} = \frac{a.2^{3/2}(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$$

$$= \frac{a.2^{3/2}(1+\cos\theta)^{1/2}}{3} = \frac{a.2^{3/2}\left(2\cos^2\frac{\theta}{2}\right)^{1/2}}{3}$$

$$= \frac{a.2^{3/2}\,.\,2^{1/2}\cos\frac{\theta}{2}}{3} = \frac{4a}{3}\cos\frac{\theta}{2}.$$

Let $\angle XOP_1 = \theta, \quad \therefore \quad \angle XOP_2 = \pi + \theta$

$\therefore \qquad \rho_1 \text{ at } P_1 = \frac{4a}{3}\cos\frac{\theta}{2}$ ...(1)

Changing θ to π + θ in (1), we have

$$\rho_2 \text{ at } P_2 = \frac{4a}{3}\cos\left(\frac{\pi+\theta}{2}\right) = \frac{4a}{3}\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$= -\frac{4a}{3}\sin\frac{\theta}{2}$ ...(2)

Squaring and adding (1) and (2), we have

$$\rho_1^{\,2} + \rho_2^{\,2} = \frac{16a^2}{9}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) = \frac{16a^2}{9}.$$

**Example 9.** *Find the radius of curvature at the point (p, r) of the ellipse*

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}.$$

**Sol.** Equation of curve is $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$

or $\frac{a^2b^2}{p^2} = b^2 + a^2 - r^2$ *i.e.,* $r^2 = a^2 + b^2 - \frac{a^2b^2}{p^2}$

Differentiating w.r.t. $p$, $2r\frac{dr}{dp} = \frac{2a^2b^2}{p^3} \quad \therefore \quad \rho = r\frac{dr}{dp} = \frac{a^2b^2}{p^3}.$

**Example 10.** *If φ be angle which the radius vector of the curve r = f(θ), makes with the tangent, prove that* $\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)$, *where ρ is the radius of curvature.*

*Apply this result to show that ρ = a/2 for circle r = a cos θ.*

**Sol.** We know that $\psi = \theta + \phi$ [*Refer to Fig. of Art. 6*]

∴ Differentiating w.r.t. $s$, we have

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left[1 + \frac{d\phi}{d\theta}\right]$$

or $\frac{1}{\rho} = \frac{\sin\phi}{r}\left[1 + \frac{d\phi}{d\theta}\right]$ $\left[\because \ \sin\phi = r\frac{d\theta}{ds}\right]$

$\therefore \qquad \frac{r}{\rho} = \sin\phi\left[1 + \frac{d\phi}{d\theta}\right]$ ...(*i*)

Also equation of circle is $r = a \cos\theta$

Differentiating, $r_1 = \dfrac{dr}{d\theta} = -a\sin\theta$

$\therefore \quad \tan\phi = r\dfrac{d\theta}{dr} = \dfrac{r}{r_1} = \dfrac{a\cos\theta}{-a\sin\theta} = -\cot\theta = \tan\left(\dfrac{\pi}{2}+\theta\right)$

NOTES

$\therefore \quad \phi = \dfrac{\pi}{2}+\theta$ and $\dfrac{d\phi}{d\theta} = 1$

But from (*i*), $\rho = \dfrac{r\ \mathrm{cosec}\ \phi}{(1+d\phi/d\theta)} = \dfrac{a\cos\theta\ \mathrm{cosec}\ (\frac{1}{2}\pi+\theta)}{1+1}$

$= \dfrac{a}{2}\cos\theta\ .\ \sec\theta = \dfrac{a}{2}.$

## EXERCISE B

**1.** Show that for the curve $r = a(1 + \cos\theta)$, the radius of curvature

$$\rho = \frac{4a}{3}\cos\frac{\theta}{2} \quad \text{and} \quad \frac{\rho^2}{r} = \frac{8a}{9}.$$

**2.** Show that the radius of the curvature at any point of the cardioid $r = a(1 - \cos\theta)$ is $\dfrac{2}{3}\sqrt{2ar}$ and prove that $\dfrac{\rho^2}{r}$ is constant.

**3.** If $\rho_1, \rho_2$ be the radii of curvature at the extremities of any chord through the pole of the cardioid $r = a(1 - \cos\theta)$, show that $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$.

**4.** Find the radius of curvature at any point of the curve $r = a\cos n\theta$ and show that at the point where $r = a$, its value is $\dfrac{a}{1+n^2}$.

**5.** Find the radius of curvature of the curve $r = a\sin n\theta$ at the pole.

**6.** Find the radius of curvature at any point $(r, \theta)$ of the following curves :

(*i*) $r^m = a^m \sin m\theta$ (*ii*) $r^2\cos 2\theta = a^2$

(*iii*) $\theta = \dfrac{\sqrt{r^2 - a^2}}{a} - \cos^{-1}\dfrac{a}{r}$ (*iv*) $r = ae^{\theta\cot\alpha}$

(*v*) $\dfrac{2a}{r} = 1 + \cos\theta.$

**7.** Find the radius of curvature at any point $(p, r)$ on the following curves :

(*i*) $p^2 = ar$ (*ii*) $r^2 = a^2 - b^2 + \dfrac{a^2b^2}{p^2}$ (*iii*) $2ap^2 = r^3$ (*iv*) $pa^2 = r^3$.

**8.** Find the radius of curvature for the curves :

(*i*) $p = a(1 + \sin\psi)$ (*ii*) $p^2 = a^2\cos^2\psi + b^2\sin^2\psi$.

**9.** Find the radius of curvature to the curve $r = a(1 + \cos\theta)$ at the points where tangent is parallel to the initial line.

[**Hint.** Firstly do Q. 1 ; and then use $\theta + \phi = 180°$.]

**10.** Show that the radius of curvature of the lemniscate $r^2 = a^2\cos 2\theta$ at the point where the tangent is parallel to $x$-axis is $\dfrac{\sqrt{2}\ .\ a}{3}$.

**11.** Prove that in the curve $r^2 = a^2 \sin 2\theta$

(*i*) The curvature varies as the radius vector.

(*ii*) The tangent turns three times as fast as the radius vector.

[**Hint.** Prove that $\psi = 3\theta$]

(*iii*) Also for the same curve find the points at which radii vectors are perpendicular to the tangents and find the radii of curvature at these points.

### Answers

**4.** $\dfrac{a(\cos^2 n\theta + n^2 \sin^2 n\theta)^{3/2}}{(1+n^2)\cos^2 n\theta + 2n^2 \sin^2 n\theta}$ **5.** $\dfrac{na}{2}$

**6.** (*i*) $\dfrac{a^m}{(m+1)\,r^{m-1}}$ (*ii*) $\dfrac{r^3}{a^2}$ (*iii*) $\sqrt{r^2 - a^2}$ (*iv*) $r \,.\, \text{cosec}\, \alpha$

(*v*) $2r\sqrt{\dfrac{r}{a}}$ **7.** (*i*) $\dfrac{2r^{3/2}}{\sqrt{a}}$ (*ii*) $\dfrac{a^2b^2}{p^3}$ (*iii*) $\frac{2}{3}\sqrt{2ar}$

(*iv*) $\dfrac{a^2}{3r}$ **8.** (*i*) $a$ (*ii*) $\dfrac{a^2b^2}{p^3}$ **9.** $\dfrac{2a}{\sqrt{3}}$

**11.** (*iii*) $\left(\pm a, \dfrac{\pi}{4}\right); \dfrac{a}{3}$.

## RADIUS OF CURVATURE AT THE ORIGIN

When the curve passes through the origin, the following methods may be used for finding the radius of curvature at the origin.

(*i*) **Method of direct substitution.** Calculate the values of $y_1$ and $y_2$ at origin, and then substitute these values direct in the formula,

$$\rho = \frac{(1 + {y_1}^2)^{3/2}}{y_2}$$

(*ii*) **Method of expansion.** The above method very often fails or becomes very labourious. So let us obtain the values of $y_1(0)$ and $y_2(0)$ by the following method :

Let $y = f(x)$ be the equation of the curve. Since it passes through origin, therefore $f(0) = 0$.

$\therefore$ By Maclaurin's Expansion,

$$y = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

*i.e.,*
$$y = xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

*i.e.,*
$$y = px + \frac{1}{2!} qx^2 + \frac{1}{3!} rx^3 + \ldots\ldots \qquad \ldots(1)$$

where $p = f'(0) = y_1(0)$ ; $q = f''(0) = y_2(0)$ etc.

Diff. (1) w.r.t. $x$, $y_1 = p + \dfrac{2qx}{2!} + \dfrac{3rx^2}{3!} + \ldots\ldots$

Again diff. w.r.t. $x$, $y_2 = \dfrac{2q}{2!} + \dfrac{6rx}{3!} + \ldots$

$\therefore$ At the origin $y_1 = p$ (Putting $x = 0$ in $y_1$)

and $y_2 = \dfrac{2q}{2!} = q$

NOTES

Putting these values of $y_1$ and $y_2$ in $\rho = \dfrac{(1+{y_1}^2)^{3/2}}{y_2}$, we have

$$\rho = \frac{(1+\mathbf{p}^2)^{3/2}}{\mathbf{q}}.$$

**Remark. To find values of p and q.**

Substituting the value of $y = px + \dfrac{qx^2}{2!} + \dfrac{rx^3}{3!} + .....$ in the equation of the curve and then equate the coefficients of the like powers of $x$ in the identity thus obtained. This way we will get the values of $p$ and $q$.

(*iii*) **Newton's method.** (*a*) **If a curve passes through the origin, and axis of x is the tangent at the origin, then**

$$\rho \text{ at the origin} = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^2}{2y}.$$

(*b*) **If a curve passes through the origin and axis of y is the tangent there, then radius of curvature at the origin.**

$$= \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{y^2}{2x}.$$

(*a*) Since axis of $x$ is the tangent at the origin, therefore its slope

$$y_1(0) = \left(\frac{dy}{dx}\right)_{(0,\,0)} = 0$$

Now $\dfrac{x^2}{2y}$ is of the indeterminate form $\dfrac{0}{0}$, as $x \to 0$, $y \to 0$.

$\therefore$ By Hospital's rule

$$\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^2}{2y} = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{2x}{2y_1} = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x}{y_1} \qquad \left[\text{Again of form } \frac{0}{0}, \because \; y_1(0) = 0\right]$$

$$= \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{1}{y_2} = \frac{1}{y_2(0)} \qquad ...(i)$$

But $\rho$ at origin $= \dfrac{[1+{y_1}^2(0)]^{3/2}}{y_2(0)} = \dfrac{(1+0)^{3/2}}{y_2(0)} = \dfrac{1}{y_2(0)}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*), $\rho$ (*at origin*) $= \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \dfrac{x^2}{2y}$.

(*b*) Proof is similar to part (*a*) and left as an exercise for the students. (Interchange the letters $x$ and $y$ in the above proof).

**Cor. Curvature at the origin when polar equation of the curve is given.** If the initial line is the tangent at the pole (origin), then

$$\rho \text{ (at the origin or pole)} = \underset{x \to 0}{\text{Lt}} \frac{x^2}{2y} = \underset{\theta \to 0}{\text{Lt}} \frac{r^2 \cos^2 \theta}{2r \sin \theta} = \underset{\substack{\theta \to 0 \\ r \to 0}}{\text{Lt}} \left(\frac{r}{2\theta} \cdot \frac{\theta}{\sin \theta} \cdot \cos^2 \theta\right)$$

NOTES

$$= \underset{\substack{\theta \to 0 \\ r \to 0}}{\text{Lt}} \left( \frac{r}{2\theta} \right) \qquad \left[ \because \ \frac{\theta}{\sin\theta} \to 1, \cos\theta \to 1, \text{when } \theta \to 0 \right]$$

$$= \underset{\substack{\theta \to 0 \\ r \to 0}}{\text{Lt}} \frac{\frac{dr}{d\theta}}{2} \qquad | \text{ L's Hospital Rule}$$

$$= \textbf{Lt} \left( \frac{1}{2} \frac{dr}{d\theta} \right).$$

**Remark.** The tangents to an algebraic curve at the origin are obtained by **equating the lowest degree terms to zero.**
(See chapter on Singular Points Page ????)

## SOLVED EXAMPLES

**Example 11.** *Show that the radii of curvature of the curve* $y^2 = x^2 \dfrac{(a+x)}{(a-x)}$ *at the origin are* $a\sqrt{2}$.

**Sol.** Equation of the curve is $\quad y^2(a - x) = x^2 (a + x) \qquad \ldots(i)$

*i.e.,* $\quad y^2(a - x) - x^2(a + x) = 0$

Equating to zero the lowest degree terms, we get

$$a(y^2 - x^2) = 0 \quad \therefore \quad y = \pm x \text{ are the tangents at origin.}$$

$\therefore$ Newton's Method is not applicable here.

Let $\quad y = px + \dfrac{1}{2!} qx^2 + \dfrac{1}{3!} rx^3 + \ldots\ldots$

Substituting this in the equation of the curve, we get

$$(a - x)(px + \tfrac{1}{2} qx^2 + \tfrac{1}{6} rx^3 + \ldots\ldots)^2 = x^2(a + x)$$

Equating coefficients of $x^2$ and $x^3$ on both sides of the above identity, we have

$$ap^2 = a \quad \ldots(i) \qquad \text{and} \qquad apq - p^2 = 1 \qquad \ldots(ii)$$

From (*i*), $\quad p^2 = 1 \quad$ or $\quad p = \pm 1$

When $p = 1$, then (*ii*) gives $aq - 1 = 1$ or $q = \dfrac{2}{a}$

$$\therefore \quad \rho \text{ at the origin} = \frac{(1+p^2)^{3/2}}{q} = \frac{(1+1)^{3/2}}{2/a} = a\sqrt{2}$$

And when $p = -1$, then (*ii*) gives $-aq - 1 = 1$ or $q = -2/a$

$$\therefore \quad \rho \text{ at the origin} = \frac{(1+1)^{3/2}}{-2/a} = -a\sqrt{2}.$$

**Example 12.** *Find the radius of curvature at the origin for the curve*

$$x^3 - 2x^2y + 3xy^2 - 4y^3 + 5x^2 - 6xy + 7y^2 - 8y = 0.$$

**Sol.** The curve passes through the origin. Equating to zero the lowest degree terms, we find $y = 0$, *i.e.,* $x$-axis is the tangent at the origin.

$$\therefore \quad \text{By Newton's formula } \rho \text{ at } (0, 0) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^2}{2y}$$

Dividing the equation of the curve by $2y$, we get

$$x \cdot \frac{x^2}{2y} - x^2 + \frac{3}{2}xy - 2y^2 + 5\frac{x^2}{2y} - 3x + \frac{7}{2}y - 4 = 0$$

Taking limits as $x$ and $y$ both tend to zero, we have

$$5 \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^2}{2y} - 4 = 0, \text{ other terms become zero.}$$

*i.e.,* $\quad 5\rho - 4 = 0 \quad$ or $\quad \rho = \frac{4}{5}$.

## EXERCISE 3

1. Find the radius of curvature at the origin for the curve $x^3 + y^2 - 2x^2 + 6y = 0$.
2. Find the radius of curvature at the origin for the curves :
   (*i*) $2x^4 + 3y^4 + 4x^2y + xy - y^2 + 2x = 0$
   (*ii*) $2x^3 + 4x^2y + xy^2 + 5y^3 - x^2 - 2xy + y^2 + 4x = 0$
   (*iii*) $2x^4 + 4x^3 + xy^2 + 6y^3 - 3x^2 - 2xy + y^2 - 4x = 0$.
3. Show that the radius of curvature at the origin for the curve $x^3 + y^3 = 3axy$ is equal to $\frac{3a}{2}$.
   [**Hint.** Divide both sides by $2xy$.]
4. Find the radius of curvature at the origin of the following curves :
   (*i*) $y = 6x + 5x^2 + x^3$ (*ii*) $y - x = x^2 + 2xy + y^2$
   (*iii*) $a(y^2 - x^2) = x^3$.

### Answers

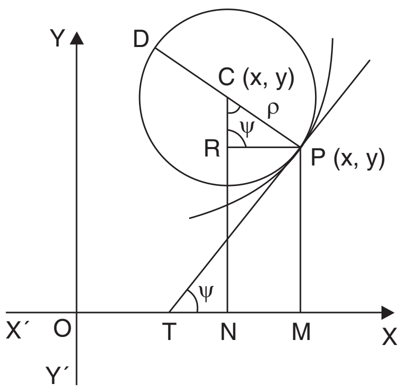
**1.** $\frac{3}{2}$ **2.** (*i*) 1 (*ii*) 2 numerically (*iii*) 2.

**4.** (*i*) $\frac{37\sqrt{37}}{10}$ (*ii*) $\frac{1}{2\sqrt{2}}$ (*iii*) $2\sqrt{2}\,a$ (numerically).

## CENTRE OF CURVATURE, CIRCLE OF CURVATURE AND EVOLUTE

(*i*) **Centre of Curvature** *for any point P of a curve is the point on the positive direction of the normal at P (i.e., the direction on the concave side of curve) at a distance* $\rho$ *from it.*

Let PD be the normal to the curve at P, and C be a point on it such that PC = $\rho$, then C is the centre of curvature of the curve at P.




(*ii*) **Evolute of a Curve.** The locus of the centres of curvature of the given curve is called the **Evolute** of the curve.

(*iii*) *The circle with its centre at the centre of curvature C and radius equal to* $\rho$ *is called the* **circle of curvature** *of the curve at the point P.*

**Remark.** Evidently the circle of curvature touches the curve at P and both the curve and the circle of curvature have the same curvature at this point.

(*iv*) *To find the co-ordinates of* **centre of curvature** *for any point* $P(x, y)$ *of curve* $y = f(x)$.

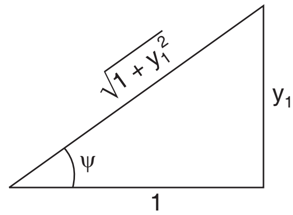
Let C(X, Y) be centre of curvature corresponding to any point $P(x, y)$ on the curve, then PC = ρ. (above Fig.). Let tangent TP make an angle ψ with positive direction of $x$-axis. Draw PM and CN perpendicular on $x$-axis, and draw PR perpendicular to CN. Then

$$\angle PCN = 90^\circ - \angle CPR = 90^\circ - (90^\circ - \angle RPT) = \angle RPT = \angle PTX = \psi$$

$$\therefore \quad X = ON = OM - NM = OM - RP = x - CP\sin\psi = x - \rho\sin\psi \quad \ldots(1)$$

and $$Y = NC = NR + RC = MP + RC = y + CP\cos\psi = y + \rho\cos\psi \quad \ldots(2)$$

But we know that $\tan\psi = y_1$




$$\therefore \quad \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}} \quad \text{and} \quad \cos\psi = \frac{1}{\sqrt{1+y_1^2}}$$

Also $$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

Hence substituting these values in (1) and (2), we get

$$\mathbf{X = x - \frac{y_1(1+y_1^2)}{y_2}, \ Y = y + \frac{1+y_1^2}{y_2}}.$$

**Cor. 1.** We have already proved that

$$\mathbf{X = x - \rho\sin\psi, \text{ and } Y = y + \rho\cos\psi.}$$

Since $x, y, \rho, \psi$ depend upon $s$ ; therefore the above equations may be treated as **parametric equations of the evolute.**

**Cor. 2.** $$\rho = \frac{ds}{d\psi}, \ \sin\psi = \frac{dy}{ds} \text{ and } \cos\psi = \frac{dx}{ds}$$

∴ Substituting these values, we get

$$X = x - \frac{ds}{d\psi}\cdot\frac{dy}{ds} = x - \frac{dy}{d\psi} \quad \text{and} \quad Y = y + \frac{ds}{d\psi}\cdot\frac{dx}{ds} = y + \frac{dx}{d\psi},$$

co-ordinates of centre of curvature in another form.

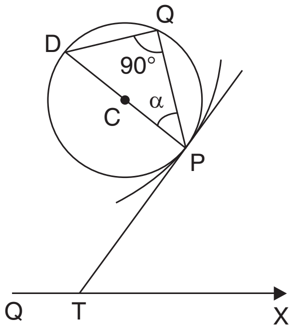
**Cor. 3.** To find the **equation of the circle of curvature** at a given point of the curve.

Let (α, β) be the centre of curvature and ρ be radius of curvature at the given point. Then equation of circle of curvature at the given point is

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2.$$

## CHORD OF CURVATURE

*The length intercepted by the circle of curvature of the curve at P, on a straight line drawn through P in any given direction is called* **chord of curvature** *through P in that direction.* Thus, if the chord of curvature PQ, makes angle α, with the normal PCD, then its length PQ is given by




$$PQ = PD\cos\alpha \quad [\because \ \angle DQP, \text{ being a semi-circle is art. angle}]$$

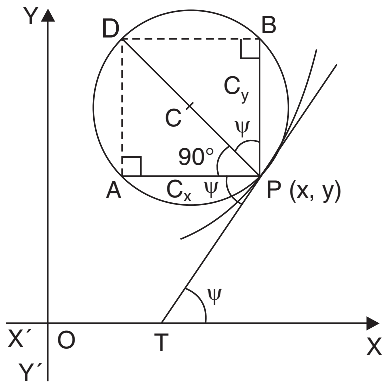
$$= 2\rho\cos\alpha$$

We now proceed to find the *length of chord of curvature* in some particular cases.

NOTES

### (*i*) Cartesian Co-ordinates.

*Chords of curvature parallel to the axes.*




Let the tangent at P make angle ψ with *x*-axis, then the chord of curvature PA, parallel to *x*-axis, makes an angle 90° – ψ with the normal PCD, and chord of curvature PB, parallel to *y*-axis makes angle ψ with the normal PCD.

∴ $C_x$ = *length of the chord of curvature PA, parallel to x-axis.*

$= PD \cos (90° - \psi)$

$= \mathbf{2\rho \sin \psi}$

$$= \frac{2\,.\,(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{1+y_1^2}}$$

$$= \frac{\mathbf{2\,y_1\,(1+y_1^2)}}{\mathbf{y_2}}$$

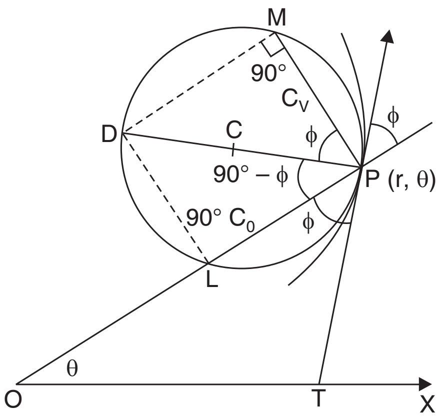
and $C_y$ = *length of the chord of curvature PB, parallel to y-axis*

$$= PD \cos \psi = \mathbf{2\rho \cos \psi} = \frac{2(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{1+y_1^2}}$$

$$= \frac{\mathbf{2(1+y_1^2)}}{\mathbf{y_2}}.$$

### (*ii*) Polar Co-ordinates

Chord of curvature through the pole and perpendicular to the radius vector.




PL, the chord of curvature through pole O, makes angle of 90° – φ, with PCD, the normal to the curve at P, and PM the chord of curvature ⊥ to the radius vector OP, makes angle φ with the normal PCD.

∴ $\mathbf{C_0}$ = *length of chord of curvature PL through the pole.*

$= PD \cos (90° - \phi)$

$= \mathbf{2\rho \sin \phi}$

$$= \frac{2(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \cdot \frac{r}{\sqrt{r^2+r_1^2}}$$

$$= \frac{2r(r^2+r_1^2)}{r^2+2r_1^2-rr_2}$$

and $\mathbf{C_p}$ = *length of chord of curvature PM ⊥ to radius vector*

$$= PD \cos \phi = \mathbf{2\rho \cos \phi} = \frac{2\,.\,(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \cdot \frac{r_1}{\sqrt{r^2+r_1^2}} = \frac{2r_1(r^2+r_1^2)}{r^2+2r_1^2-rr_2}.$$

### (*iii*) Pedal Equations

When the pedal equation $p = f(r)$ of the curve is given, then

$$C_0 = \textit{length of chord of curvature through pole along radius vector.}$$
$$= PD\cos(90^\circ - \phi) = 2\rho\sin\phi \quad \ldots(1)$$

But $\quad \rho = r\,.\,\dfrac{dr}{dp}$ and $\sin\phi = \dfrac{p}{r}$

$\therefore$ Substituting in (1), we get

$$C_0 = 2r\frac{dr}{dp}\cdot\frac{p}{r} = 2p\,.\,\frac{dr}{dp}.$$

Also from equation $p = f(r)$, we have

$$\frac{dp}{dr} = f'(r), \quad \text{and} \quad \sin\phi = \frac{p}{r} = \frac{f(r)}{r}$$

$\therefore$ From (1), $\quad C_0 = 2\rho\sin\phi = 2\,.\,r\dfrac{dr}{dp}\,.\sin\phi = 2r\,.\,\dfrac{1}{f'(r)}\cdot\dfrac{f(r)}{r} = \dfrac{2f(r)}{f'(r)}.$

Also $\quad C_p = \textit{length of chord} \perp \textit{to the radius vector}$

$$= DP\cos\phi = 2\rho\cos\phi = 2\,.\,r\frac{dr}{dp}\cdot\frac{\sqrt{r^2-p^2}}{r}$$

$$\left[\because \sin\phi = \frac{p}{r}, \quad \therefore \cos\phi = \frac{\sqrt{r^2-p^2}}{r}\right]$$

$$= 2\,.\sqrt{r^2-p^2}\,\frac{dr}{dp}.$$

## SOLVED EXAMPLES

**Example 13.** *Find the coordinates of the centre of curvature at any point (x, y) of the parabola $y^2 = 4ax$. Also find the equation of the evolute of the parabola.*

**Sol.** Equation of the parabola is $y^2 = 4ax$ the parametric equations of this parabola are

$$x = at^2, \qquad y = 2at.$$

$$\therefore \quad \frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a$$

$$y_1 = \frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} = \frac{2a}{2at} = \frac{1}{t}$$

Again differentiating w.r.t. $x$,

$$y_2 = -\frac{1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2}\cdot\frac{1}{2at} = -\frac{1}{2at^3}$$

$\therefore$ Co-ordinates of centre of curvature at any point $(x, y)$ of the parabola are

$$X = x - y_1\frac{(1+y_1^{\,2})}{y_2} = at^2 - \frac{1}{t}\frac{\left(1+\dfrac{1}{t^2}\right)}{-\dfrac{1}{2at^3}} = at^2 + 2at^2\left(1+\frac{1}{t^2}\right)$$

or $\quad X = 3at^2 + 2a \quad ...(1)$

and $\quad Y = y + \dfrac{1}{y_2}(1 + y_1^2) = 2at - 2at^3\left(1 + \dfrac{1}{t^2}\right)$

$\quad = 2at - 2at^3 - 2at = -2at^3 = -2at^3 \quad ...(2)$

From (1) and (2),

The coordinates of centre of curvature at any point $(x, y) = (at^2, 2at)$ of the parabola are $(3at^2 + 2a, -2at^3)$.

Note, we know that evolute of a curve is the locus of centres of curvature.

In fact, equations (1) and (2) are parametric equations of the evolute of parabola.

To get cartesian equation of evolute of parabola, let us eliminate $t$ from (1) and (2).

Form (1), $\quad t^2 = \dfrac{X - 2a}{3a} \quad ...(3)$

From (2), $\quad t^3 = -\dfrac{Y}{2a} \quad ...(4)$

Now cubing (3), squaring (4) and equating the two values of $t^6$, we get

$$\left(\frac{X - 2a}{3a}\right)^3 = \frac{Y^2}{4a^2} \quad \text{or} \quad 27aY^2 = 4(X - 2a)^3$$

Changing X to $x$ and Y to $y$,

we get $\quad 27ay^2 = 4(x - 2a)^3,$

which is the equation of the evolute of the parabola.

**Example 14.** *Find the coordinates of the centre of curvature of ellipse*

*$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $x = a\cos\theta$, $y = b\sin\theta$. Hence show that the equation of its evolute is*

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$

**Sol.** Let (X, Y) be the coordinates of the centre of curvature at point 'θ' of the given ellipse

$$x = a\cos\theta, \qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = b\cos\theta$$

$\therefore \quad y_1 = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta} = \dfrac{b\cos\theta}{-a\sin\theta} = -\dfrac{b}{a}\cot\theta$

and $\quad y_2 = \dfrac{d}{dx}\left(-\dfrac{b}{a}\cot\theta\right) = \dfrac{d}{d\theta}\left(-\dfrac{b}{a}\cot\theta\right).\dfrac{d\theta}{dx}$

$$= \frac{b}{a}\text{cosec}^2\theta \, . \, \frac{1}{-a\sin\theta} = -\frac{b}{a^2}\text{cosec}^3\theta$$

$\therefore \quad X = x - y_1\dfrac{(1 + y_1^2)}{y_2} = a\cos\theta - \dfrac{-\dfrac{b}{a}\cot\theta\left(1 + \dfrac{b^2}{a^2}\cot^2\theta\right)}{-\dfrac{b}{a^2}\text{cosec}^3\theta}$

$$= a\cos\theta - a\sin^3\theta\,\frac{\cos\theta}{\sin\theta}\left(1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

**NOTES**

$$= a\cos\theta - a\cos\theta \,.\, \sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta\,(1-\sin^2\theta) - \frac{b^2}{a}\cos^3\theta = a\cos^3\theta - \frac{b^2}{a}\cos^3\theta$$

$$= \frac{a^2-b^2}{a}\cos^3\theta \qquad \ldots(i)$$

and
$$Y = y + \frac{(1+y_1^2)}{y_2} = b\sin\theta + \frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)}{\frac{-b}{a^2}\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - \frac{a^2\sin^3\theta}{b}\left(1+\frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

$$= b\sin\theta\,(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta = b\sin^3\theta - \frac{a^2}{b}\sin^3\theta$$

$$= -\frac{a^2-b^2}{b}\sin^3\theta \qquad \ldots(ii)$$

(*i*) and (*ii*) give the co-ordinates (X,Y) of the centre of curvature of the ellipse. To find the equation of its evolute, we eliminate θ between (*i*) and (*ii*).

From (*i*), $a\text{X} = (a^2 - b^2)\cos^3\theta$, $\therefore\ (a\text{X})^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta$

and from (*ii*), $b\text{Y} = -(a^2-b^2)\sin^3\theta$ $\therefore\ (b\text{Y})^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta$

Adding, we get $(a\text{X})^{2/3} + (b\text{Y})^{2/3} = (a^2-b^2)^{2/3}$

Changing X to $x$ and Y to $y$, we get

$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$, which is the equation of the evolute of the ellipse.

**Example 15.** *If $C_x$ and $C_y$ be the chords of curvature parallel to the axes of x and y respectively, at any point of the curve $y = ae^{x/a}$, prove that* $\dfrac{1}{C_x^2} + \dfrac{1}{C_y^2} = \dfrac{1}{2aC_x}$.

**Sol.** Equation of the curve is $y = ae^{x/a}$

Differentiating, $y_1 = a\,.\,e^{x/a}\,.\,\dfrac{1}{a} = e^{x/a} = \dfrac{y}{a}$ $\qquad [\because\ y = ae^{x/a}]$

and $y_2 = e^{x/a}\dfrac{1}{a} = \dfrac{y}{a}\,.\,\dfrac{1}{a} = \dfrac{y}{a^2}$

Now
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{y^2}{a^2}\right)^{3/2}}{\frac{y}{a^2}} = \frac{(a^2+y^2)^{3/2}}{ay}$$

Also $\tan\psi = \dfrac{dy}{dx} = \dfrac{y}{a}$, $\therefore\ \sin\psi = \dfrac{y}{\sqrt{a^2+y^2}}$ and $\cos\psi = \dfrac{a}{\sqrt{a^2+y^2}}$

Now
$$C_x = 2\rho\sin\psi = 2\,.\,\frac{(a^2+y^2)^{3/2}}{ay}\,.\,\frac{y}{\sqrt{a^2+y^2}} = \frac{2(a^2+y^2)}{a} \qquad \ldots(i)$$

and
$$C_y = 2\rho\cos\psi = 2\,\frac{(a^2+y^2)^{3/2}}{ay}\,.\,\frac{a}{\sqrt{a^2+y^2}} = \frac{2(a^2+y^2)}{y}$$

$$\therefore \quad \frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{a^2}{4(a^2+y^2)^2} + \frac{y^2}{4(a^2+y^2)^2} = \frac{1}{4(a^2+y^2)}$$

$$= \frac{1}{2a}\left[\frac{a}{2(a^2+y^2)}\right] = \frac{1}{2a\,.\,C_x}. \qquad [\because \text{ of } (i)]$$

## EXERCISE D

**1.** Find the centre of curvature for the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the folium $x^3 + y^3 = 3axy$.

**2.** Find the evolute of the parabola $y^2 = 4ax$.
[**Hint.** It is example 1.]

**3.** Find the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
[**Hint.** It is example 2.]

**4.** Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
[**Hint.** Take the parametric equations of the hyperbola as $x = a \sec \theta$, $y = b \tan \theta$.
Prove that centre of curvature (X, Y) is $\left(\frac{a^2+b^2}{a}\sec^3\theta, -\frac{(a^2+b^2)}{b}\tan^3\theta\right)$.
Then use $\sec^2\theta - \tan^2\theta = 1$.]

**5.** Define evolute of a curve and show that the evolute of the hypocycloid

$$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } (x+y)^{2/3} + (x-y)^{2/3} = 2\,.\,a^{2/3}.$$

[**Hint.** Parametric equations of this curve are $x = a\cos^3\theta$, $y = a \sin^3\theta$.]

**6.** Find the centre of curvature for any point $(x, y)$ on the rectangular hyperbola $xy = c^2$ and find the equation of its evolute.

**7.** (*a*) Show that the equation of the circle of curvature at point $\left(\frac{a}{4}, \frac{a}{4}\right)$ on the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \text{ is } \left[x - \frac{3}{4}a\right]^2 + \left[y - \frac{3}{4}a\right]^2 = \frac{1}{2}a^2.$$

(*b*) Show that the circle of curvature at the origin of the parabola

$$y = mx + \frac{x^2}{a} \text{ is } x^2 + y^2 = a(1+m^2)(y - mx).$$

**8.** Find the centre of curvature C for any point P on the catenary $y = c\cosh\left(\frac{x}{c}\right)$ and show that PC = PG where G is the point of intersection of normal at P with $x$-axis.
[**Hint.** PG = Length of the normal = $y\sqrt{1+y_1^2}$.]

**9.** Show that the evolute of the tractrix $x = c \cos t + c \log \tan \frac{1}{2} t$, $y = c \sin t$ is the catenary $y = c\cosh\frac{x}{c}$.

**10.** Prove that the chord of curvature parallel to $y$-axis for the curve $y = a \log \sec \frac{x}{a}$ is of constant length.

**11.** If $C_x$, $C_y$ be chords of curvatures parallel to the axis of $x$ and $y$ respectively at any point of the curve $y = c\cosh\frac{x}{c}$ ; prove that $4c^2(C_x^2 + C_y^2) = C_y^4$.

**12.** (*a*) Find the chords of curvature through the pole for the curves :

(*i*) $r = ae^{m\theta}$ (*ii*) $r^n = a^n \cos n\theta$.

(*b*) Show that for the curve $p = ae^{br}$, the chord of curvature through the pole is of constant length.

**13.** (*a*) Show that the length of the chord of curvature through the pole of the curve $r = a\,(1 + \cos\theta)$ is $\dfrac{4r}{3}$.

(*b*) If $C_o$ and $C_p$ denote lengths of chords of curvature of the cardioide $r = a(1 + \cos\theta)$, along and perpendicular to radius vector through any point, show that $3(C_o^2 + C_p^2) = 8aC_0$.

**14.** Show that at any point on the equiangular spiral $r = ae^{\theta\cot\alpha}$, $\rho = r\,\text{cosec}\,\alpha$, and show that radius of curvature subtends a right angle at the pole.

**15.** Prove that the points on the curve $r = f(\theta)$, the circle of curvature at which passes through the pole (origin) are given by the equation $f(\theta) + f''(\theta) = 0$.

## Answers

**1.** $\left(\dfrac{21}{16}a, \dfrac{21}{16}a\right)$.

**2.** $27ay^2 = 4(x - 2a)^3$.

**3.** $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$

**4.** $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$.

**6.** $\left(\dfrac{3x}{2} + \dfrac{c^4}{2x^3}, \dfrac{3c^2}{2x} + \dfrac{x^3}{2c^2}\right)$; $(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}$.

**8.** $\left(x - c\cosh\dfrac{x}{c}\sinh\dfrac{x}{c}, y + c\cosh\dfrac{x}{c}\right)$.

**12.** (*a*) (*i*) $2r$ (*ii*) $\dfrac{2r}{n+1}$.

# 11. ASYMPTOTES

NOTES

**STRUCTURE**



## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Asymptotes Parallel to Axes of Coordinates
- Asymptotes Parallel to Coordinate Axes for Algebraic Curve $f(x, y$
- Working Rule for Finding Polar Asymptotes

## BRANCHES OF A CURVE

Let us consider the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solving for $y$, we get

$$y = b\sqrt{1 - \frac{x^2}{a^2}} \quad \dots (i) \quad \text{or} \quad y = -b\sqrt{1 - \frac{x^2}{a^2}} \quad \dots (ii)$$

These two equations (*i*) and (*ii*) represent explicity the two branches of the ellipse which in ordinary language we call the upper and lower half of the ellipse. It is to be noticed that the ellipse lies inside the rectangle whose sides are $x = \pm a$ and $y = \pm b$.

Thus we see that both the branches of the ellipse lie wholly within a *finite part* of the $x$-$y$ plane and we say, therefore, that both branches of the ellipse are **finite.**

Now let us consider the rectangular hyperbola $x^2 - y^2 = a^2$.

Solving for $y$, we get $y = \sqrt{x^2 - a^2}$ or $y = -\sqrt{x^2 - a^2}$.

If $x \to \pm \infty$, $y$ also tends of $\pm \infty$. In this case both the branches *extend to infinity* and are said to be the **infinite branches** of the rectangular hyperbola.

Again, let the curve $x^2y^2 = x^2 - y^2$, be considered. Solving for $y$, we have

$$y = \frac{x}{\sqrt{x^2+1}} \quad \text{or} \quad y = -\frac{x}{\sqrt{x^2+1}}.$$

Now as $x \to \infty$, $y \to 1$ and as $x \to -\infty$, $y \to -1$, along the first branch. Also in the case of second branch, as $x \to \infty$, $y \to -1$ and as $x \to -\infty$, $y \to 1$. Here both the branches are infinite, and $x$ is capable of taking arbitrarily large values whereas $y$ remains finite.

We are already familiar with the symbols $x \to \pm \infty$ and $y \to \pm \infty$. But what does P $\to \infty$ stand for, P being a point on an infinite branch of curve. We give the following definition :

**Def. 1.** *A point P(x, y) on an infinite branch of a curve is said to tend to infinity along the curve if either x or y or both tend to* $+\infty$ *or* $-\infty$ *as P travels along the branch of the curve.*

## ASYMPTOTE

### Def. 1

*A straight line, at a finite distance from the origin is said to be a* (**rectilinear**) **asymptote** *to an infinite branch of a curve, if the perpendicular distance of a point P on that branch from the straight line tends to zero, as P tends to infinity along the branch.*

### Def. 2

*Another definition* of an asymptote is as follows :

A (**rectilinear**) **asymptote** *to an infinite branch of a curve is the limiting position of the tangent whose point of contact tends to infinity along the branch, but which itself remains at a finite distance from the origin.*

### Def. 3

If a st. line cuts a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity is called an asymptote to the curve.

**Note 1.** In the discussion which follows, we drop the word *rectilinear* and use simply *asymptote* to mean rectilinear asymptote, unless otherwise stated.

**2.** The asymptotes may be parallel to either $x$-axis or the $y$-axis and accordingly they are called *horizontal* and *vertical* asymptotes. If an asymptote is not parallel to $y$-axis ; it is called and *oblique asymptote*.

## ASYMPTOTES PARALLEL TO AXES OF COORDINATES

(*i*) *To find the asymptote parallel to y-axis, of the curve* $y = f(x)$.

Let the line $\qquad x = k \qquad$ ...(1)

parallel to $y$-axis be an asymptote to the curve $y = f(x)$.

NOTES

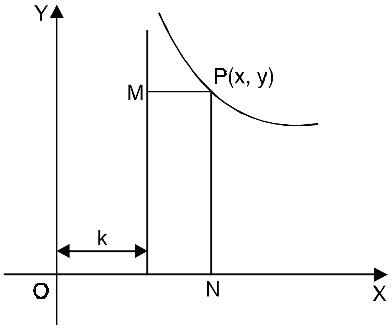
Then it is required to determine the value of $k$.

Let PM be the distance of a point $P(x, y)$ on the curve from the line (1), then $PM = (x - k)$*

Now by definition of asymptote, if line (1) is an asymptote to the curve, then $PM \to 0$, as $P \to \infty$.

$\therefore$ As $P \to \infty$, $PM = (x - k) \to 0$ or $x \to k$.

If $P(x, y)$ tends to infinity as $x \to k$, only $y$ coordinate tends to infinity (*i.e.*, $+\infty$ or $-\infty$), and this gives




$$\lim_{y \to \infty} x = k \quad i.e., \quad x \to k \text{ as } y \to \infty \quad \text{...(2)}$$

*Hence to find the asymptotes parallel to y-axis, we find from the given equation, the definite values $k_1, k_2, k_3$, ...... to which x tends, as $y \to +\infty$ or $-\infty$. Then $x = k_1$, $x = k_2$, $x = k_3$, ...... are the asymptotes parallel to y-axis.*

(*ii*) *Asymptotes parallel to x-axis*. Proceeding as above, we arrive at the following method of finding asymptotes parallel to $x$-axis. *From the given equation, find the definite values $d_1, d_2, d_3$, ...... to which y tends as $x \to +\infty$ or $-\infty$; then $y = d_1$, $y = d_2$, $y = d_3$, ....... are asymptotes parallel to the x-axis.*

## ASYMPTOTES PARALLEL TO COORDINATE AXES FOR ALGEBRAIC CURVE f(x, y) = 0

**Case** (*i*) *Asymptotes parallel to y-axis* :

Now let the equation of the algebraic curve $f(x, y) = 0$, after arranging in descending powers of $y$, be $y^n\phi_0(x) + y^{n-1}\phi_1(x) + y^{n-2}\phi_2(x) + ...... + \phi_n(x) = 0$ ...(3)

where $\phi_0(x), \phi_1(x), \phi_2(x), ..... \phi_n(x)$, are polynomials in $x$.

Dividing equation (3) throughout by $y^n$ ($\because$ $y \to \infty$), we get

$$\phi_0(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + ........ + \frac{1}{y^n}\phi_n(x) = 0$$

Taking limits as $y \to \infty$ and $x \to k$, we get [From (2)]

$$\phi_0(k) = 0 \quad \text{...(4)}$$

Eliminating $k$ from (1) and (4), combined equation of asymptotes parallel to $y$-axis is $\phi_0(x) = 0$.

Thus we arrive at the following *working* rule for finding asymptotes || to $y$-axis.

**Rule.** *To find asymptotes parallel to y-axis, equate to zero the coefficient of the highest power of y, present in the given equation of the curve. Resolve it now into real linear factors.*

If the coefficient of the highest power of $y$ is either a *constant* or *not resolvable into real linear factors* ; then there are no asymptotes parallel to $y$-axis.

**Case.** (*ii*) *Asymptotes parallel to x-axis for the curve f(x, y) = 0.*

By interchanging $y$ and $x$ in Case (*i*), we arrive at the following rule.

**Rule.** *To find asymptotes parallel to x-axis, equate to zero the coefficient of the highest power of x, present in given equation of the curve.* Resolve L.H.S. into real linear factors.

*Perpendicular distance of the point $(x_1, y_1)$ from the straight line

$ax + by + c = 0$ is $\dfrac{(ax_1 + by_1 + c)}{\sqrt{a^2 + b^2}}$.

If the coefficient of the highest power of $x$ is either a *constant* or *not resolvable into real linear factors,* there are *no* asymptotes parallel to $x$-axis.

**Note.** In the chapter on asymptotes, care should be taken to write the equation of the curve s.t. R.H.S. is zero.

NOTES

## SOLVED EXAMPLES

**Example 1.** *Find the asymptotes, parallel to the axes for the curve :*

(*i*) $x^2y^2 = a^2(x^2 + y^2)$ (*ii*) $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1.$

**Sol.** (*i*) The equation of the curve is (Making R.H.S. zero) $x^2y^2 - a^2(x^2 + y^2) = 0.$

Equating to zero the coefficient of $x^2$, the highest power of $x$, we get $y^2 - a^2 = 0$ or $y = \pm a$, which are the asymptotes parallel to $x$-axis.

Again equating to zero the coefficient of $y^2$, the highest power of $y$, we get $x^2 - a^2 = 0$ or $x = \pm a$, which are the asymptotes parallel to $y$-axis.

(*ii*) The equation of the curve is $\frac{a^2}{x^2} - \frac{b^2}{y^2} = 1$

Multiplying by L.C.M. $= x^2y^2$.

or $$a^2y^2 - b^2x^2 = x^2y^2 \quad \text{or} \quad x^2y^2 + b^2x^2 - a^2y^2 = 0$$

Equating to zero the coefficient of $x^2$, the highest power of $x$, we get

$$y^2 + b^2 = 0 \quad \text{or} \quad y^2 = -b^2 \quad \text{or} \quad y = \pm ib,$$

which gives imaginary values of $y$, and therefore there is no asymptote parallel to $x$-axis.

Again equating to zero the coefficient of $y^2$, the highest power of $y$, we get

$$x^2 - a^2 = 0 \quad \text{or} \quad x = \pm a,$$

which are the asymptotes parallel to $y$-axis.

## EXERCISE A

*Find the asymptotes, parallel to axes, of the following curves :*

**1.** $x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0.$

**2.** $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1.$

**3.** $xy^3 - x^3 = a(x^2 + y^2).$

**4.** $x^2y^2 - y^2 - 2 = 0.$

**5.** $y^3 - xy^2 = x^2 + 1.$

**6.** $y = x(x - 2)(x - 3).$

### Answers

**1.** $y = \pm a, x = \pm a$ **2.** $y = \pm b$ ; $x = \pm a$ **3.** $x = 0$

**4.** $y = 0, x = \pm 1$ **5.** No. **6.** No.

## OBLIQUE ASYMPTOTES

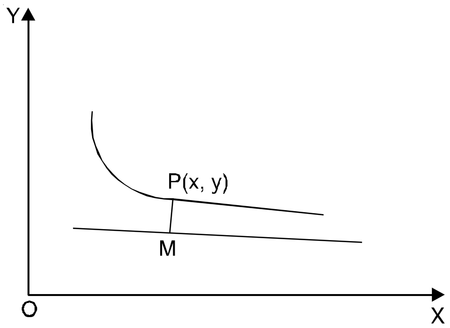
**If $y = mx + c$ is an oblique asymptote to any curve $f(x, y) = 0$ ; then $m = \underset{x \to \infty}{\text{Lt}} \frac{y}{x}$ and $c = \underset{x \to \infty}{\text{Lt}} (y - mx)$.**

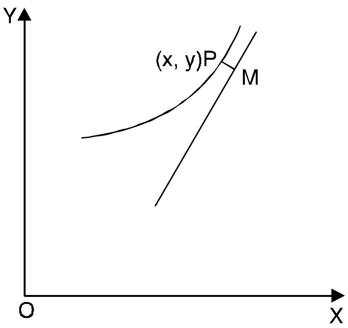
NOTES

Let the straight line, $y = mx + c$ *i.e.*, $y - mx - c = 0$ ...(*i*)

be an oblique asymptote of the curve $f(x, y) = 0$, not parallel to $y$-axis so that $m$ and $c$ are both finite.

Let P($x$, $y$) be any point on the infinite branch of the curve, which corresponds to asymptote (*i*). If $p$ = PM be perpendicular distance of point P from straight line (*i*), then




$$p = \pm \frac{y - mx - c}{\sqrt{1+m^2}} \quad \text{or} \quad y - mx = c \pm p\sqrt{1+m^2} \qquad ...(ii)$$

dividing both side by $x$, $\dfrac{y}{x} = m + \dfrac{c}{x} \pm \dfrac{p}{x}\sqrt{1+m^2}$

Since strainght line (*i*) is an asymptote, it follows, from definition, that $p \to 0$, as $x \to \infty$. Taking limits

$$\therefore \qquad \underset{x\to\infty}{\text{Lt}} \frac{y}{x} = \underset{x\to\infty}{\text{Lt}} \left[ m + \frac{c}{x} \pm \frac{p}{x}\sqrt{1+m^2} \right] = m$$

Hence $\qquad \underset{x\to\infty}{\text{Lt}} \dfrac{y}{x} = m \qquad ...(iii)$

Again from (*ii*), $\underset{x\to\infty}{\text{Lt}} (y - mx) = \underset{x\to\infty}{\text{Lt}} \left[ c \pm p\sqrt{1+m^2} \right]$

$$= c \pm \sqrt{1+m^2} \underset{x\to\infty}{\text{Lt}} p = c + 0 = c$$

Hence $\qquad \underset{x\to\infty}{\text{Lt}} (y - mx) = c \qquad ...(iv)$

**Note 1.** Thus from (*iii*) and (*iv*), $m$ and $c$ are determined, and putting these values in (*i*), the oblique asymptotes can be found out.

**2.** The above article requires $m$ and $c$ to be finite. In particular $m$ may be zero and therefore the asymptotes parallel to $x$-axis can also be found by using this article. Thus by the method of this article all the asymptotes of a given curve can be determined, excepting those which are parallel to $y$-axis.

## OBLIQUE ASYMPTOTES OF THE GENERAL RATIONAL ALGEBRAIC EQUATION

Let the general algebraic equation of $n$th degree be

$$(a_0 y^n + a_1 y^{n-1} x + a_2 y^{n-2} x^2 + ... + a_{n-1} y x^{n-1} + a_n x^n) + (b_1 y^{n-1} + b_2 y^{n-2} . x$$
$$+ ... + b_{n-1} y x^{n-2} + b_n x^{n-1}) + ... + (l_{n-1} y + l_n x) + k_n = 0 \quad ...(i)$$

NOTES

which can be written in the form

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+ .... + x\phi_1\left(\frac{y}{x}\right)+\phi_0\left(\frac{y}{x}\right)=0 \quad ...(ii)$$

where $\phi_r\left(\frac{y}{x}\right)$ is a polynomial in $\frac{y}{x}$ of degree $r$.

[We know that a homogeneous expression of degree $n$ in $x$ and $y$ can be written as $x^n\ \phi_n\left(\frac{y}{x}\right)$]

Let the straight line $\qquad y = mx + c \qquad ...(iii)$

be an asymptote of curve (*ii*), where $m$ and $c$ are finite ;

then $\qquad m = \underset{x\to\infty}{\text{Lt}}\ \frac{y}{x}\ ;\ c = \underset{x\to\infty}{\text{Lt}}\ (y - mx) \qquad$ [By Art. 4]

**To find m,** divide both sides of (*ii*) by $x^n$, we get

$$\phi_n\left(\frac{y}{x}\right)+\frac{1}{x}\cdot\phi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\phi_{n-2}\left(\frac{y}{x}\right)+............+......=0$$

Proceeding to limit as $x \to \infty$, so that $\underset{x\to\infty}{\text{Lt}}\ \frac{y}{x} = m$, we get

$$\boldsymbol{\phi_n\ (m) = 0} \quad ...(iv)$$

which gives the slopes of the asymptotes.

**To find c,** corresponding to a value of $m$.

Let $y - mx = p$, so that as $x \to \infty$, $p \to c$ $\qquad \left[\because\ c = \underset{x\to\infty}{\text{Lt}}\ (y - mx)\right]$

Then $\frac{y}{x} = m + \frac{p}{x}$. Putting this value for $\frac{y}{x}$ in (*ii*), we get

$$x^n\phi_n\left(m+\frac{p}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{p}{x}\right)+......+x\phi_1\left(m+\frac{p}{x}\right)+\phi_0\left(m+\frac{p}{x}\right)=0$$

Now expanding each term by Taylor's theorem, we have

$$x^n\left[\phi_n(m)+\frac{p}{x}\phi_n{}'(m)+\frac{p^2}{2!\,x^2}\phi''_n(m)+....\right]+x^{n-1}\left[\phi_{n-1}(m)+\frac{p}{x}\phi'_{n-1}(m)+....\right]$$

$$+\ x^{n-2}\left[\phi_{n-2}(m)+\frac{p}{x}\cdot\phi'_{n-2}(m)+....\right]+...=0 \quad ...(v)$$

But $\phi_n\ (m) = 0$, from (*iv*). Putting $\phi_n(m) = 0$ and arranging the remaining terms in (*v*) is descending powers of $x$, we get

$$x^{n-1}\ [p\phi_n{}'(m)+\phi_{n-1}(m)]+x^{n-2}\left[\frac{p^2}{2!}\cdot\phi''_n(m)+p\cdot\phi'_{n-1}(m)+\phi_{n-2}(m)\right]+....=0 \quad ...(vi)$$

Dividing throughout by $x^{n-1}$, we have

$$p\phi_n{}'(m)+\phi_{n-1}(m)+\frac{1}{x}\left[\frac{p^2}{2!}\cdot\phi''_n(m)+p\cdot\phi'_{n-1}(m)+\phi_{n-2}(m)\right]+....=0$$

Now proceeding to limits as $x \to \infty$, so that $p \to c$, we have

$$\boldsymbol{c\phi'_n\ (m) + \phi_{n-1}\ (m) = 0} \quad ..(vii)$$

$\therefore$ If $\phi_n{}'(m) \neq 0$, which will be case, if equation $\phi_n(m) = 0$, has no repeated roots, then we have $c = -\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$.

$\therefore$ If $m_1, m_2, m_3, \ldots\ldots$ be the *non-repeated* roots of $\phi_n(m) = 0$, and $c_1, c_2, c_3, \ldots\ldots$ are the corresponding values of $c$ determined from (*vii*), then $y = m_1x + c_1$, $y = m_2x + c_2$, $y = m_3x + c_3$ .... are the asymptotes.

*Exceptional Case.* When $\phi_n'(m) = 0$, but $\phi_{n-1}(m) \neq 0$, the finite value of $c$ cannot be determined from (*vii*), and there is *no asymptote* in this case.

**Case of parallel asymptotes.** If $\phi_n'(m) = 0$ and $\phi_{n-1}(m) = 0$, which is generally there when two roots of $\phi_n(m) = 0$ *i.e.*, two values of $m$ given by $\phi_n(m) = 0$ are equal, then equation (*vii*) becomes an identify and in this case, the equation (*vi*) reduces to

$$x^{n-2}\left[\frac{p^2}{2!}\phi''_n(m) + p\phi'_{n-1}(m) + \phi_{n-2}(m)\right] + x^{n-3}\,[+ \ldots] + \ldots = 0$$

Dividing throughout by $x^{n-2}$, we get

$$\frac{p^2}{2!}\,\phi_n''(m) + p\phi'_{n-1}(m) + \phi_{n-2}(m) + \frac{1}{x}\,[\ldots\ldots] + \ldots\ldots = 0$$

Now proceeding to limits as $x \to \infty$, so that $p \to c$, we have

$$\mathbf{\frac{c^2}{2!}\,\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0}$$

which gives two values say $c', c''$ *for* $c$, provided $\phi_n''(m) \neq 0$.

Thus we have two asymptotes

$y = mx + c'$ ; $y = mx + c''$, corresponding to a given slope $m$. These are evidently *parallel*.

The above discussion leads us to the following **working rule for finding oblique asymptotes of an algebraic curve of nth degree.**

(*i*) *Find the polynomial* $\phi_n(m)$, *which can be obtained by putting* $x = 1, y = m$ *in the highest degree terms of the given equation of the curve.*

(*ii*) *Put* $\phi_n(m)$ *equal to zero, solve for m, and let the roots be* $m_1, m_2, m_3, \ldots\ldots$

(*iii*) *Find the polynomial* $\phi_{n-1}(m)$ *by putting* $x = 1, y = m$ *in next lower degree terms of the given equation. Similarly, polynomial* $\phi_{n-2}(m)$ *can be found out by putting* $x = 1, y = m$ *in the next lower degree terms of the equation, and so on.*

(*iv*) *Find the values* $c_1, c_2, c_3, \ldots\ldots$ *corresponding to values* $m_1, m_2, m_3, \ldots\ldots$ *by using the relation* $c = -\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$. [*provided* $\phi_n'(m) \neq 0$]

(*v*) *The required asymptotes are* $y = m_1x + c_1$, $y = m_2x + c_2$, $y = m_3x + c_3$ .......

(*vi*) *If* $\phi_n'(m) = 0$ *for some value of m, but* $\phi_{n-1}(m) \neq 0$, *then corresponding to that value of m there is no asymptote.*

(*vii*) *If* $\phi_n'(m) = 0 = \phi_{n-1}(m)$, *for some value of m (i.e., two roots of* $\phi_n(m) = 0$ *are equal), then the values of c are found from the equation*

$$\frac{c^2}{2!}\,\phi_n''(m) + c.\phi'_{n-1}(m) + \phi_{n-2}(m) = 0.$$

*This gives two values of c, and therefore there are two parallel asymptotes corresponding to this value of m.*

(*viii*) *If the three roots of* $\phi_n(m) = 0$ *are equal ; [i.e.,* $\phi_n''(m) = 0$, $\phi'_{n-1}(m) = 0$, $\phi_{n-2}(m) = 0$*], then values of c corresponding to that value of m are obtained from the equation*

$$\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi''_{n-1}(m) + c\phi'_{n-2}(m) + \phi_{n-3}(m) = 0.$$

**Note 1.** Asymptote corresponding to $m = 0$, as a root of $\phi_n(m) = 0$, is parallel to $x$-axis, and are directly obtained by method of Article 3 case (*ii*).

**2.** When it is required to find the asymptotes of a curve, it is advisable to first find the asymptotes parallel to axes, if there be any, and *then* search for the oblique asymptotes.



**Example 1.** *Find all the asymptotes of the following curves :*

(*a*) $x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0.$

(*b*) $x^3 + 4x^2y + 4xy^2 + 5x^2 + 15xy + 10y^2 - 2y + 1 = 0.$

**Sol.** (*a*) The equation of the curve is

$$x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0 \qquad ...(i)$$

Since the coefficients of $x^3$ and $y^3$, the highest degree terms in $x$ and $y$, are constant, $\therefore$ there are no asymptotes parallel to $x$-axis or $y$-axis. *Now to find oblique asymptotes* ; putting $x = 1$, $y = m$ in the third and second degree terms in (*i*), we get

$$\phi_3(m) = 1 + 2m - m^2 - 2m^3 \text{ and } \phi_2(m) = 3m + 3m^2$$

The slopes of asymptotes are the roots of $\phi_n(m) = 0$ *i.e.*, $\phi_3(m) = 0$,

*i.e.*, of $\quad 1 + 2m - m^2 - 2m^3 = 0$ or $2m^3 + m^2 - 2m - 1 = 0.$

$m = 1$ is a root of this equation by inspection. Using synthetic division

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 1 | – 2 | – 1 |
| | | 2 | 3 | 1 |
| | 2 | 3 | 1 | 0 |

$\therefore$ The reduced equation is $2m^2 + 3m + 1 = 0$

Solving for $m$, we get $\qquad m = -1, \dfrac{-1}{2}$

$\therefore \qquad m = -1, 1, -\dfrac{1}{2}$

Also $\qquad \phi_3'(m) = 2 - 2m - 6m^2 = -(6m^2 + 2m - 2)$

Now $c$ is given by $\qquad c\phi_3'(m) + \phi_2(m) = 0 \qquad$ | $\mathbf{c\phi_n'(m) + \phi_{n-1}(m) = 0}$

$$\therefore \qquad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{3m + 3m^2}{-(6m^2 + 2m - 2)} = \frac{3m + 3m^2}{6m^2 + 2m - 2}$$

When $m = -1$, $\qquad c = \dfrac{-3 + 3}{6 - 2 - 2} = \dfrac{0}{2} = 0$

When $m = 1$, $\qquad c = \dfrac{3 + 3}{6 + 2 - 2} = \dfrac{6}{6} = 1$

and when $m = -\dfrac{1}{2}$, $\qquad c = \dfrac{-\dfrac{3}{2} + \dfrac{3}{4}}{\dfrac{3}{2} - 1 - 2} = \dfrac{1}{2}.$

Putting these values of $m$ and $c$ in $y = mx + c$, the corresponding asymptotes are

$$y = -x + 0, \qquad y = x + 1 \qquad \text{and} \qquad y = -\frac{1}{2}x + \frac{1}{2}$$

or $\qquad x + y = 0, \qquad x - y + 1 = 0 \qquad$ and $\qquad x + 2y - 1 = 0.$

(*b*) The equation of the curve is

$$x^3 + 4x^2y + 4xy^2 + 5x^2 + 15xy + 10y^2 - 2y + 1 = 0 \qquad ...(i)$$

Since the coefficient of $x^3$, the highest degree term in $x$, is constant, $\therefore$ there is no asymptote parallel to $x$-axis. Again equating to zero the coefficient of $y^2$, the highest degree term in $y$, the asymptote parallel to $y$-axis is given by the equation

$$4x + 10 = 0 \quad \text{or} \quad 2x + 5 = 0$$

Now to find oblique asymptotes, putting $x = 1$, $y = m$ in the third and second degree terms in the equation of the curve, we get

$$\phi_3(m) = 1 + 4m + 4m^2,$$

and
$$\phi_2(m) = 5 + 15m + 10m^2$$

Slopes of asymptotes are given by the roots of equation $\phi_3(m) = 0$ *i.e.*, by roots of

$$1 + 4m + 4m^2 = 0 \quad \text{or} \quad (1 + 2m)^2 = 0$$

$$\therefore \qquad m = -\frac{1}{2}, -\frac{1}{2} \qquad \text{(Equal)}$$

$\therefore$ $c$ is now given by the equation

$$\frac{c^2}{2!}\phi_n''(m) + c\,\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$$

Here
$$\frac{1}{2!}c^2\,\phi_3''(m) + c\phi'_2(m) + \phi_1(m) = 0 \qquad \ldots(ii)$$

Now $\phi_3'(m) = 4 + 8m$, $\phi_3''(m) = 8$, $\phi_2'(m) = 15 + 20m$ and $\phi_1(m) = -2m$

$\therefore$ Equation (*ii*) becomes $\frac{1}{2!}c^2 \,.\, 8 + c(15 + 20m) - 2m = 0$

For $m = -\frac{1}{2}$, it becomes $4c^2 + c(15 - 10) + 1 = 0$

or
$$4c^2 + 5c + 1 = 0 \quad \text{or} \quad (4c + 1)(c + 1) = 0$$

$$\therefore \qquad c = -1, \quad c = -\frac{1}{4}$$

Hence the corresponding asymptotes are

$$y = -\frac{1}{2}x - 1 \qquad \text{and} \qquad y = -\frac{1}{2}x - \frac{1}{4}$$

or
$$x + 2y + 2 = 0 \qquad \text{and} \qquad 2x + 4y + 1 = 0.$$

**Example 3.** *Show that the parabola* $y^2 - 4ax = 0$ *has no asymptotes.*

**Sol.** Since the coefficient of $y^2$, the highest degree term in $y$, in the equation of the parabola is a constant, $\therefore$ there are no asymptotes parallel to $y$-axis. Again, since the coefficient of $x$, the highest degree term in $x$ is also a constant, $\therefore$ there is no asymptote parallel to $x$-axis.

For *finding oblique asymptotes* putting $x = 1$, $y = m$ in the second degree and first degree terms in the equation of the parabola, we get

$$\phi_2(m) = m^2 \quad \text{and} \quad \phi_1(m) = -4a.$$

Slopes of asymptotes are given by the roots of equation $\phi_2(m) = 0$ *i.e.*, by the equation $m^2 = 0$ or $m = 0, 0$

But asymptotes (if any) corresponding to $m = 0$ are parallel to $x$-axis. But it has been proved above that there are no asymptotes parallel to $x$-axis.

Hence the curve has no oblique asymptote.

Since the parabola has neither oblique asymptotes nor asymptotes parallel to the axes,

$\therefore$ it has no asymptotes.

**Example 4.** *Find all the asymptotes of the curve* $(x + y)^2 (x + y + 2) = x + 9y - 2$.

**Sol.** The equation of the curve is

$$(x + y)^2 (x + y + 2) = x + 9y - 2$$

or
$$(x + y)^3 + 2(x + y)^2 - x - 9y + 2 = 0 \qquad \ldots(i)$$

Since the coefficients of $x^3$ and $y^3$, the highest degree terms in $x$ and $y$ are constants, therefore there are no asymptotes parallel to $x$-axis or $y$-axis.

NOTES

Now *to find oblique asymptotes* ; putting $x = 1$, $y = m$ in the third, second, first degree terms and constant terms in (*i*), we have

| | |
|---|---|
| $\phi_3(m) = (1 + m)^3$ | $\phi_3'(m) = 3(1 + m)^2$ |
| $\phi_2(m) = 2(1 + m)^2$ | $\phi_3''(m) = 6(1 + m)$ |
| $\phi_1(m) = -1 - 9m$ | $\phi_3'''(m) = 6$ |
| $\phi_0(m) = 2$ | $\phi_2'(m) = 4(1 + m)$ |
| | $\phi_2''(m) = 4$ |
| | $\phi_1'(m) = -9$ |

Slopes of the oblique asymptotes are given by $\phi_3(m) = 0$

*i.e.*, by $(1 + m)^3 = 0 \quad \therefore \quad m = -1, -1, -1$

For **these three equal values** of $m = -1$, values of $c$ are given by

$$\mathbf{\frac{c^3}{3!}\phi_3'''(m) + \frac{c^2}{2!}\phi_2''(m) + c\phi_1'(m) + \phi_0(m) = 0}$$

or $$\frac{c^3}{6} . 6 + \frac{c^2}{2}(4) + c(-9) + 2 = 0 \quad \text{or} \quad c^3 + 2c^2 - 9c + 2 = 0$$

$c = 2$ is a root of this equation by inspection.

Using synthetic division,

| 2 | 1 | 2 | – 9 | 2 |
|---|---|---|---|---|
| | | 2 | 8 | – 2 |
| | 1 | 4 | –1 | 0 |

$\therefore$ The reduced equation is $c^2 + 4c - 1 = 0$

Solving for $c$, $$c = \frac{-4 \pm \sqrt{16 + 4}}{2} = \frac{-4 \pm 2\sqrt{5}}{2}$$

or $$c = -2 \pm \sqrt{5}$$

$\therefore$ For $m = -1$, the three different values of $c$ are

$$2, -2 + \sqrt{5} \text{ and } -2 - \sqrt{5}$$

$\therefore$ Equations of the three asymptotes are

$$y = mx + c$$

*i.e.*, $$y = -x + 2,\ y = -x - 2 + \sqrt{5}$$

and $$y = -x - 2 - \sqrt{5}.$$

## EXERCISE B

*Find all the asymptotes of the following curves (Q. No. 1—16) :*

**1.** $x^2y^2 = a^2(x^2 + y^2)$ **2.** $xy^2 = 4a^2(2a - x)$ **3.** $x^3 + y^3 = 3axy$

**4.** (*a*) $x^3 + 3xy^2 - x^2y - 3y^3 + x^2 - 2xy + 3y^2 + 4x + 7 = 0$

(*b*) $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$

**5.** $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$

**6.** $x^3 + 4x^2y + 5xy^2 + 2y^3 + 2x^2 + 4xy + 2y^2 - x - 9y + 1 = 0$

**7.** $y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x + 1 = 0$

**8.** (*a*) $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0$

(*b*) $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0$

(*c*) $x^3 - 2x^2y + xy^2 + x^2 - xy + 2 = 0$

NOTES

**9.** (*a*) $x(y-x)^2 = x(y-x)+2$

(*b*) $4x^2(y-x)+y(y-2)(x-y)=4x+4y-7$

**10.** $x^2(x-y)^2+a^2(x^2-y^2)=a^2xy$

**11.** $(x-1)(x-2)(x+y)+x^2+x+1=0$

**12.** $x^3+3x^2y-4y^3-x+y+3=0$

**13.** $x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$

**14.** $x^2y^2(x^2-y^2)^2=(x^2+y^2)^3$

**15.** $(x+y)^2(x^2+xy+y^2)=a^2x^2+a^3(y-x)$

**16.** $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)-2a^2(x-2y)(x+y)=0.$

**17.** Prove that the asymptotes of the curve $x^2y^2=a^2(x^2+y^2)$ are the sides of a square.

**18.** Show that the asymptotes of $x^2y^2-a^2(x^2+y^2)-a^3(x+y)+a^4=0$ form a square, through two of whose vertices the curve passes.

**19.** Find rectangular asymptotes of the curve $y=\dfrac{x}{x^2-1}$.

[**Hint.** Rectangular Asymptotes $\Rightarrow$ Asymptotes at right angles to each other. Cross-mutiplying $y(x^2-1)=x$.]

**20.** Find the asymptotes of the curve $(x+y+1)(x^2+y^2-xy)-3xy+x^2+y^2+2x-3y+5=0.$

[**Hint.** Simplifying $(x+y)(x^2+y^2-xy)+(x^2+y^2-xy)-3xy+x^2+y^2+2x-3y+5=0$

or $$(x+y)(x^2+y^2-xy)+2x^2+2y^2-4xy+2x-3y+5=0.]$$

**21.** Find the asymptotes of the curve $(x+y+1)^2(x^2+y^2-xy)+3xy-7x^2-2y^2-7x+8=0.$

**Remark.** The above equation can be written as

$$[(x+y)^2+2(x+y)+1]\,(x^2+y^2-xy)+3xy-7x^2-2y^2-7x+8=0$$

or $$(x+y)^2\,(x^2+y^2-xy)+2(x+y)\,(x^2+y^2-xy)+x^2+y^2-xy+3xy-7x^2-2y^2-7x+8=0$$

or $$(x+y)^2\,(x^2+y^2-xy)+2(x+y)(x^2+y^2-xy)-6x^2-y^2+2xy-7x+8=0.$$

## Answers

**1.** $x=\pm a,\ y=\pm a.$ **2.** $x=0.$ **3.** $x+y+a=0.$

**4.** (*a*) $2x-2y+1=0.$ (*b*) $x-y=0,\ x-2y=0,\ x-3y=0.$

**5.** $y=-x-2,\ y=x-1,\ y=2x.$ **6.** $x+2y+2=0,\ x+y\pm 2\sqrt{2}=0.$

**7.** $x-y=0,\ 2x-y+1=0,\quad 2x-y+2=0.$

**8.** (*a*) $2x+y=0,\ 2x+y+1=0,\quad x-y=0$

(*b*) $x=0,\quad x+y=0,\quad x+y=1.$

(*c*) $x=0,\quad y-x=0,\quad y-x=1.$

**9.** (*a*) $x=0,\quad y=x,\ y=x+1.$

(*b*) $x-y=0,\ 2x-y+1=0$ and $2x+y-1=0.$

**10.** $x=\pm a,\ y=x\pm a.$ **11.** $x=1,\quad x=2,\quad x+y+1=0.$

**12.** $y-x=0,\quad 2y+x-1=0$ and $2y+x+1=0.$

**13.** $x+y-1=0,\ 2x-2y+(3+\sqrt{5})=0,\quad 2x-2y+(3-\sqrt{5})=0.$

**14.** $x\pm y=\pm\sqrt{2},\quad x=\pm 1,\quad y=\pm 1.$ **15.** $x+y\pm a=0.$

**16.** $x=y+a,\quad x=y+2a,\quad 2y=x+14a,\quad 3y=x-13a.$

**19.** $x=\pm 1,\quad y=0.$ **20.** $3x+3y+8=0.$

**21.** $x+y-1=0$ and $x+y+3=0.$

## TOTAL NUMBER OF ASYMPTOTES

NOTES

*To show that a curve of degree n can never have more than n asymptotes.*

Consider the equation of algebraic curve of $n$th degree of the form

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots = 0,$$

where $\phi_r\left(\frac{y}{x}\right)$ is a polynomial in $\frac{y}{x}$ of degree $r$.

The slopes of oblique asymptotes are given by $\phi_n(m) = 0$, which is an equation of degree $n$ and thus has *at the most n* real roots.

Since for each value of $m$, we have, in general, one value of $c$ determined by the equation $c\phi_n'(m) + \phi_{n-1}(m) = 0$, $\therefore$ it follows that a *curve of degree n has,in general, at the most n asymptotes*. In case the curve has one or more asymptotes parallel to $y$-axis, then the degree of equation $\phi_n(m) = 0$, is smaller than $n$ by at least the same number.

Further when equation determining $c$ is an equation of second degree ; then $\phi_n'(m) = 0$ and $\therefore$ it follows that $\phi_n(m) = 0$, has two equal roots. Thus, there are two values of $c$ corresponding to two equal roots and there would be at the most $(n - 2)$ other asymptotes corresponding to the remaining roots. Hence it follows that a curve of degree $n$ can never have more than $n$ asymptotes.

## ASYMPTOTES BY INSPECTION

*If equation of a curve of degree n, is of the form $F_n + F_{n-2} = 0$ where $F_n$ is of degree n [i.e., contains terms of degree n and may also contain lower degree terms], and $F_{n-2}$ is of degree $(n - 2)$ at the most, then every linear factor of $F_n$ equated to zero, will be an asymptote, provided no two linear factors of $F_n$ are either coincident or differ by a constant.*

Let $(ax + by + c)$ be a non-repeated factor of $F_n$ and let

$F_n = (ax + by + c)\, F_{n-1}$ where $F_{n-1}$ is of degree $(n - 1)$.

The asymptote parallel to $ax + by + c = 0$, is given by

$$ax + by + c + \underset{\substack{x \to \infty \\ \frac{y}{x} \to -\frac{a}{b}}}{\text{Lt}} \frac{F_{n-2}}{F_{n-1}} = 0$$

Now to find the limit $\frac{F_{n-2}}{F_{n-1}}$, the numerator as well as denominator is divided by $x^{n-1}$ and it would be seen that $\frac{1}{x}$ appears as a factor

$$\therefore \qquad \text{Lt}\ \frac{F_{n-2}}{F_{n-1}} \to 0, \text{ as } x \to \infty.$$

Hence $ax + by + c = 0$, is an asymptote.

**Cor.** If the equation of the curve is of the form $U_1 . U_2 . U_3 \ldots\ldots U_n + F_{n-2} = 0$ where $U_1, U_2, U_3, \ldots\ldots, U_n$ are linear factor, then the curve has $U_1 = 0, U_2 = 0, U_3 = 0, \ldots, U_n = 0$ as asymptotes provided no two of the factors are coincident or differ by a constant.

**Note. Conversely, a curve having straight lines $U_1 = 0$, $U_2 = 0$ ...... $U_n = 0$, as asymptotes, must have an equation of the form $U_1 . U_2 . U_3 \ldots\ldots\ldots\ldots U_n + F_{n-2} = 0$.**

NOTES

## SOLVED EXAMPLES

**Example 5.** (*a*) *Find the asymptotes of* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

(*b*) *Find asymptotes of the curve* $xy(x^2 - y^2) + 2x^2 + 2y^2 + 1 = 0$.

**Sol.** (*a*) The equation of the curve is $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) - 1 = 0$

This is of the form $F_n + F_{n-2} = 0$ [$n = 2$], where $F_2$ can be split up into non-repeated linear factors.

Thus the asymptotes are given by $F_2 = 0$

or $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad \left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 0$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} - \frac{y}{b} = 0,$$

are the required asymptotes.

(*b*) The equation of the curve is $xy(x^2 - y^2) + 2x^2 + 2y^2 + 1 = 0$

This is of the form $F_n + F_{n-2} = 0$ $(n = 4)$.

The linear factors of $F_4 = xy(x^2 - y^2)$ are $x$, $y$, $(x - y)$ and $(x + y)$.

Since none of them is repeated, the four asymptotes of the curve are

$$x = 0, \quad y = 0, \quad x - y = 0 \quad \text{and} \quad x + y = 0.$$

## INTERSECTIONS OF A CURVE WITH ITS ASYMPTOTES

*To prove that any asymptote of an algebraic curve of the nth degree cuts the curve in (n – 2) points.*

Let $$y = mx + c \quad \ldots(i)$$

be an asymptote of the algebraic curve

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right) + \ldots = 0 \quad \ldots(ii)$$

To find points of intersections of asymptote (*i*) and curve (*ii*) let us solve them for $x$ and $y$.

Putting value of $y$ from (*i*) in (*ii*) $\left(i.e., \text{Putting } \frac{y}{x} = m + \frac{c}{x}\right)$

the abscissae of the points of intersection are the roots of the equation

$$x^n\phi_n\left(m + \frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m + \frac{c}{x}\right) + x^{n-2}\phi_{n-2}\left(m + \frac{c}{x}\right) + \ldots\ldots = 0$$

Expanding by Taylor's theorem, we have

$$x^n\left[\phi_n(m) + \frac{c}{x}\phi_n'(m) + \frac{1}{2!}\frac{c^2}{x^2}.\phi_n''(m) + \ldots\right] + x^{n-1}\left[\phi_{n-1}(m) + \frac{c}{x}\phi'_{n-1}(m) + \ldots\right]$$
$$+ x^{n-2}[\phi_{n-2}(m) + \ldots] + \ldots = 0$$

Or arranging the terms according to descending power of $x$, we get

$$x^n\phi_n(m) + x^{n-1}[c\,.\,\phi_n'(m) + \phi_{n-1}(m)]$$
$$+ x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m)\right] + \ldots = 0 \quad \ldots(iii)$$

NOTES

Since ($i$) is an asymptote of ($ii$), $\therefore$ $m$ and $c$ are given by

$$\phi_n(m) = 0 \quad \text{and} \quad c\phi_n'(m) + \phi_{n-1}(m) = 0$$

Substituting these values in ($iii$), it reduces to

$$x^{n-2}\left[\frac{c^2}{2!}\phi_n''(m) + c\phi_{n-1}'(m) + \phi_{n-2}(m)\right] + x^{n-3}[...] + ... = 0$$

which is an equation of $(n - 2)$ th degree and determines $(n - 2)$ values of $x$.

Hence ($i$) cuts ($ii$) in $(n - 2)$ points.

**Cor. 1.** *If a curve of nth degree has n asymptotes, then they cut the curve in $n(n - 2)$ points.*

**Cor. 2.** *If the equation of a curve of nth degree can be put in the form $F_n + F_{n-2} = 0$, where $F_{n-2}$ is of degree $(n - 2)$ at the most and $F_n$ consists of n non-repeated linear factors, then the $n(n - 2)$ points of intersection of the curve and its asymptotes lie on the curve $F_{n-2} = 0$.*

**Proof.** The joint equation of $n$ asymptote of curve is $F_n = 0$, and the $n(n - 2)$ point of intersection of the curve and its asymptote satisfy the two equations $F_n + F_{n-2} = 0$ and $F_n = 0$, simultaneously.

$\therefore$ They also satisfy the equation, $(F_n + F_{n-2}) - F_n = 0$ *i.e.*, $F_{n-2} = 0$

Hence the result.

For example :

($i$) The asymptotes of a *cubic* curve, cut the curve in 3(3 – 2) = 3 points which lie on a curve of degree 3 – 2 = 1 *i.e.*, on a *straight line*.

($ii$) The asymptotes of a *biquadratic or quartic curve*, cut the curve in 4(4 – 2) = 8 points which lie on a curve of degree 4 – 2 = 2, *i.e.*, on a *conic*.

## METHOD TO FIND THE EQUATION OF A CURVE JOINING THE POINTS OF INTERSECTIONS OF THE GIVEN CURVE AND ITS ASYMPTOTES

**Step I.** Find the asymptotes of the given curve by the methods explained in Art. 3 and Art. 5.

**Step II.** Make the R.H.S. of the equation of each of these asymptotes as zero and multiply the L.H.S. of all these equations of the asymptotes to get the joint equation of asymptotes.

**Note.** But if the equation of the curve is of the form $F_n + F_{n-2} = 0$ as given in Art. 7 ; then there is no need of doing steps I and II ; as the joint equation of the asymptotes as explained in Art. 7 is $F_n = 0$ provided $F_n$ is the product of non-repeated linear factors.

**Step III.** The number of points of intersections of the curve and the asymptotes is $n(n - 2)$ (Cor. 1., Art. 8) where $n$ is the degree of the curve.

**Step IV.** Then equation of the required curve is L.H.S. of equation of the given curve after making R.H.S. zero + $\lambda$ (joint equation of the asymptotes after making R.H.S. zero) where $\lambda$ = 1 or – 1 or 2 or – 2 etc.

**Example 6.** *Show that the points of intersection of the curve $2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0$, and its asymptotes lie on the straight line $8x + 2y + 1 = 0$.*

NOTES

**Sol.** The equation of the curve is

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0 \quad ...(i)$$

Since the coefficients of $x^3$ and $y^3$, the highest degree terms in $x$ and $y$ are constants,

$\therefore$ the curve has no asymptote parallel either to $x$-axis or $y$-axis.

Now to find oblique asymptotes, putting $x = 1$, $y = m$ in the third and second degree terms in ($i$), we get $\phi_3(m) = 2m^3 - 2m - 4m^2 + 4$ and $\phi_2(m) = -14m + 6m^2 + 4$.

The slopes of asymptotes are the roots of $\phi_3(m) = 0$

*i.e.*, of $2m^3 - 2m - 4m^2 + 4 = 0$ or $(2m - 4)(m^2 - 1) = 0$

$\therefore$ $m = 2, 1, -1.$

Also $\phi_3'(m) = 6m^2 - 2 - 8m = 2(3m^2 - 4m - 1)$

Now $c$ is given by, $c\phi_3'(m) + \phi_2(m) = 0$

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{-14m + 6m^2 + 4}{2(3m^2 - 4m - 1)} = \frac{7m - 3m^2 - 2}{3m^2 - 4m - 1}$$

When $m = 2$, $c = \dfrac{14 - 12 - 2}{12 - 8 - 1} = 0$

When $m = 1$, $c = \dfrac{7 - 3 - 2}{3 - 4 - 1} = \dfrac{2}{-2} = -1$

and when $m = -1$, $c = \dfrac{-7 - 3 - 2}{3 + 4 - 1} = \dfrac{-12}{6} = -2.$

Putting values of $m$ and $c$ in $y = mx + c$, the corresponding asymptotes are

$y = 2x + 0$, $y = x - 1$ and $y = -x - 2$

Making R.H.S. of each equation as zero,

$2x - y = 0$, $x - y - 1 = 0$ and $x + y + 2 = 0.$

These three asymptotes will cut the curve again in $3(3 - 2) = 3$ points. The joint equation of the asymptotes is $(2x - y)(x - y - 1)(x + y + 2) = 0$

or $$2x^3 - 2xy^2 - x^2y + y^3 - 7xy + 3y^2 + 2x^2 + 2y - 4x = 0$$

Multiplying by 2 throughout,

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 4y - 8x = 0 \quad ...(ii)$$

Now equation of the given curve is

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 + 4x^2 + 6y + 1 = 0 \quad ...(iii)$$

Hence the 3 points of intersection of asymptotes ($ii$) and curve ($iii$) also lie on the curve ($ii$) – ($iii$)

*i.e.*, on $$8x + 2y + 1 = 0$$

which is the required straight line.

**Example 7.** *Find the equation of the quartic curve which has $x = 0$, $y = 0$, $y = x$, and $y = -x$, for asymptotes and which passes through (a, b) and which cuts the asymptotes in eight points that lie on the circle $x^2 + y^2 = a^2$.*

**Sol.** The equations of the asymptotes are $x = 0$, $y = 0$, $y - x = 0$ and $y + x = 0$

The joint equation of the asymptotes is

$$xy(y - x)(y + x) = 0 \quad \text{or} \quad xy(y^2 - x^2) = 0 \quad ...(i)$$

The equation of given circle is $x^2 + y^2 - a^2 = 0$ ...($ii$)

NOTES

The equation of any quartic curve whose asymptotes are given by (*i*) and whose points of intersection with its asymptotes lie on (*ii*) is given by

$$xy(y^2 - x^2) + \lambda(x^2 + y^2 - a^2) = 0 \qquad ...(iii)$$

where $\lambda$ is a constant.

This passes through the point $(a, b)$.

$$\therefore \qquad ab(b^2 - a^2) + \lambda(a^2 + b^2 - a^2) = 0 \quad \text{or} \quad \lambda = \frac{a(a^2 - b^2)}{b}$$

Putting this value of $\lambda$ in (*iii*), the equation of required curve is

$$xy(y^2 - x^2) + \frac{a(a^2 - b^2)}{b}(x^2 + y^2 - a^2) = 0$$

or

$$bxy(y^2 - x^2) + a(a^2 - b^2)(x^2 + y^2 - a^2) = 0.$$

## EXERCISE C

1. (*a*) Find the asymptotes of the curve $x^2y - xy^2 + xy + y^2 + x - y = 0$, and show that they cut the curve in three points on the straight line $x + y = 0$.

   (*b*) Find the equation of the straight line on which lie the three points of intersections of the cubic $x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.
2. Show that the asymptotes of the cubic, $x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$, cut the curve in three points which again lie on the straight line $x - y + 1 = 0$.
3. Find the asymptotes of the curve $x(x^2 - y^2) + y(3y - x) = 0$ and prove that the three points where these asymptotes cut the curve lies on $7x - 3y + 6 = 0$.
4. Show that the points of intersection of the curve $4x^3 - 2x^2y - 4xy^2 + 2y^3 + 6y^2 - 14xy + 4x^2 + 6y + 1 = 0$ and its asymptotes lie on $8x + 2y + 1 = 0$.
5. Show that the asymptotes of the curve $(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$ cut the curve again in eight points which lie on a circle of radius unity.
6. Find the asymptotes of the curve $xy(x^2 - y^2) + x^2 + y^2 = a^2$, and show that the eight points of intersection of the curve with its asymptotes lie on a circle whose centre is at the origin.
7. Show that the eight points of intersection of curve $x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$ and its asymptotes lie on a rectangular hyperbola.

   [**Hint.** A second degree curve is a **rectangular hyperbola** if its asymptotes are at right angles.]
8. Show that four asymptotes of the curve $xy(x^2 - y^2) + 25y^2 + 9x^2 - 144 = 0$ cut it again in eight points on an ellipse whose eccentricity is $\frac{4}{5}$.

   [**Hint.** Eccentricity $e$ of the ellipse is given by $b^2 = a^2(1 - e^2)$.]
9. Find the equation of the hyperbola having $x + y - 1 = 0$, and $x - y + 2 = 0$, as its asymptotes, and passing through the origin.
10. (*a*) Find the equation of the cubic, which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which passes through the points (0, 0), (1, 0) and (0, 1).

    (*b*) Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 4 = 0$ and which passes through (0, 0), (2, 0) and (0, 2).
11. Find the cubic which has the same asymptotes as the curve $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which touches the axis of $y$ at the origin and goes through the point (3, 2).

    [**Hint.** Tangent at the origin is obtained by equating the lowest degree terms to zero.]

**Answers**

**1.** (*a*) $y = 0$, $x - 1 = 0$ and $x - y + 2 = 0$ (*b*) $x + 3y = 1$

**3.** $x = 3$, $y = x + 1$, $x + y + 2 = 0$ **6.** $x = 0$, $y = 0$, $x \pm y = 0$

**9.** $(x + y - 1)(x - y + 2) + 2 = 0$

**10.** (*a*) $x^3 - 6x^2y + 11xy^2 - 6y^3 - x + 6y = 0$ (*b*) $x^3 - 6x^2y + 11xy^2 - 6y^3 - 4x + 2y = 0$

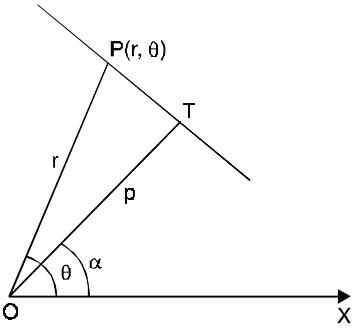
**11.** $x^3 - 6x^2y + 11xy^2 - 6y^3 - x = 0$.

NOTES

## ASYMPTOTES IN POLAR CO-ORDINATES

**Theorem 1.** *The polar equation of any line is $p = r\cos(\theta - \alpha)$ where p is the length of the perpendicular from the pole to the line and $\alpha$, is the angle which this perpendicular makes with x-axis.*

**Proof.** Let O be the pole and OX be the initial line.




Let P($r$, θ) be any point on the given line.

∴ OP = $r$ and ∠XOP = θ.

Let OT = $p$ be the length of perpendicular from pole on the given line and ∠XOT = α.

∴ ∠TOP = ∠XOP – ∠XOT = θ – α.

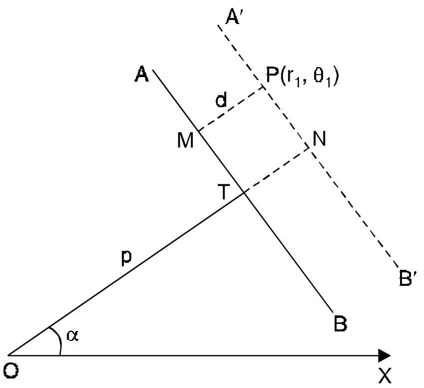
Now in ΔOPT,

$$\cos(\theta - \alpha) = \frac{OT}{OP} = \frac{p}{r}$$

or **p = r cos (θ – α)** is the required equation of the line.

**Theorem 2.** *The length of the perpendicular from a point $P(r_1, \theta_1)$ on the line $p = r\cos(\theta - \alpha)$ is $\pm(p - r_1\cos(\theta_1 - \alpha))$.*

**Proof.** Let $p = r\cos(\theta - \alpha)$ be the equation of the line AB.




∴ OT = $p$ and ∠XOT = α.

Through $P(r_1, \theta_1)$ draw a line A′ B′ parallel to AB and let PM = $d$ be the length of perpendicular from 1 on the line AB.

∴ ON = OT + TN = OT + PM = $(p + d)$

and ∠XON = α

∴ Equation of line A′ B′ is

$$p + d = r\cos(\theta - \alpha)$$

It passes through P $(r_1, \theta_1)$

∴ $p + d = r_1 \cos(\theta_1 - \alpha)$

∴ $d = -p + r_1\cos(\theta_1 - \alpha) = -(p - r_1\cos(\theta_1 - \alpha_1))$.

Similarly, by taking P below AB, we could see that

$$d = (p - r_1\cos(\theta_1 - \alpha))$$

Combining the two ; $d$ (the perpendicular distance of $P(r_1, \theta_1)$ from the line $p = r\cos(\theta - \alpha)$) is given by

$$\mathbf{d = \pm(p - r_1\cos(\theta_1 - \alpha))}$$

*Working rule for finding the perpendicular distance of a point from the line $p = r\cos(\theta - \alpha)$.*

**1.** Make R.H.S. of equation of the line as zero.

**2.** Substitute the co-ordinates of the point in L.H.S. to get ⊥ distance.

NOTES

**Polar equation of an asymptote to the curve r = f(θ) is**

$$p = r\sin(\theta_1 - \theta) \textit{ where } p = \underset{\theta\to\theta_1}{\text{Lt}} \frac{-d\theta}{du}\left(u = \frac{1}{r}\right)$$

**and $\theta_1$ is a root of the equation obtained by putting u = 0.)**

Let $\quad p = r\cos(\theta - \alpha)$ ...(*i*)

be an asymptote to the curve $r = f(\theta)$ ...(*ii*)

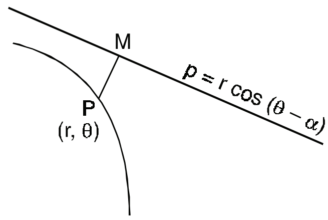
We are to determine $p$ and $\alpha$.

Let P($r$, θ) be any point on the curve (*i*).

PM, the distance of P($r$, θ) from the line (*i*)

$$= p - r\cos(\theta - \alpha) \quad ...(iii)$$

(See Working Rule Theorem 2, Art 10)




∵ Line (*i*) is an asymptote to curve (*ii*), therefore PM → 0 P($r$, θ) → ∞ along the curve.

As P($r$, θ) recedes to infinity along the curve ; $r \to \infty$ and θ tend to a finite value or values (say $\theta_1$).

(It should be noted that θ → ∞ for circular asymptotes which are beyond the scope of our learning.)

Also $\quad r \to \infty$ implies $u \to 0 \quad \left(\because\ r = \frac{1}{u}\right)$

Dividing both sides of (*iii*) by $r$ , we get

$$\frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha)$$

Taking limits as $r \to \infty$, $\theta \to \theta_1$, PM → 0, so that

$$0 = 0 - \cos(\theta_1 - \alpha) \quad \text{or} \quad \cos(\theta_1 - \alpha) = 0$$

$$\therefore \qquad \theta_1 - \alpha = \frac{\pi}{2} \quad \therefore \quad \alpha = \theta_1 - \frac{\pi}{2} \qquad ...(iv)$$

So we have determined α.

Now taking limits in (*iii*) as PM → 0, $r \to \infty$, $\theta \to \theta_1$ ; we have

$$0 = p - \underset{\substack{r\to\infty\\ \theta\to\theta_1}}{\text{Lt}}\ r\cos(\theta - \alpha)$$

or

$$p = \underset{\substack{r\to\infty\\ \theta\to\theta_1}}{\text{Lt}}\ r\cos(\theta - \alpha)$$

$$= \text{Lt}\ r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) \qquad \Big|\ \because \text{ From } (iv), \alpha = \theta_1 - \frac{\pi}{2}$$

$$= \underset{\substack{r\to\infty\\ \theta\to\theta_1}}{\text{Lt}}\ -r\sin(\theta - \theta_1) \qquad |\ \text{It is of the form } \infty\ .\ 0$$

or

$$p = \underset{\substack{r\to\infty\\ \theta\to\theta_1}}{\text{Lt}}\ -\frac{\sin(\theta - \theta_1)}{\left(\frac{1}{r}\right)} \qquad \Big|\ \text{Form } \frac{0}{0}$$

or

$$p = \underset{\substack{u\to 0\\ \theta\to\theta_1}}{\text{Lt}}\ -\frac{\sin(\theta - \theta_1)}{u} = \underset{\substack{u\to 0\\ \theta\to\theta_1}}{\text{Lt}}\ -\frac{\cos(\theta - \theta_1)}{\frac{du}{d\theta}}$$

(By L' Hospital Rule)

$$= \text{Lt}\ \frac{-1}{\frac{du}{d\theta}}\ \text{Lt}\ \cos(\theta - \theta_1) = \text{Lt}\left(-\frac{d\theta}{du}\right) \qquad ...(v)$$

$$\left[\because\ \underset{\theta\to\theta_1}{\text{Lt}}\ \cos(\theta - \theta_1) = \cos 0 = 1\right]$$

So we have determined $p$.

Putting the value of $\alpha$ from ($iv$) in ($i$), equation of the asymptote is

$$p = r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) \quad \text{or} \quad p = r\cos\left[\frac{\pi}{2} - (\theta_1 - \theta)\right]$$

or $\qquad p = r\sin(\theta_1 - \theta)$ where $p$ as given by ($v$) is $= \underset{\theta \to \theta_1}{\text{Lt}} -\dfrac{d\theta}{du}$.

## WORKING RULE FOR FINDING POLAR ASYMPTOTES

**Step I.** *Put* $r = \dfrac{1}{u}$ *in the given equation. Also change all T-ratios if any into* $\sin\theta$ *and* $\cos\theta$.

**Step II.** *Find the limit of* $\theta$ *as* $u \to 0$. *Let* $\theta_1$ *be this limit or one of the limits if more than one such limits exist.*

**Step III.** *Determine* $p = \underset{\substack{\theta \to \theta_1 \\ u \to 0}}{Lt}\left(-\dfrac{d\theta}{du}\right)$ *for value or values of* $\theta$ *obtained in Step II.*

**Step IV.** *Putting the values of* $p$ *and* $\theta_1$ *in the equation* $p = r\sin(\theta_1 - \theta)$, *we get the corresponding asymptote.*

**Note.** If $\underset{\theta \to \theta_1}{\text{Lt}}\left(-\dfrac{d\theta}{du}\right)$ does not tend to a finite limit ; then there is no asymptote corresponding to the value $\theta = \theta_1$.

### SOLVED EXAMPLES

**Example 8.** *Find the asymptotes of the curve* $r\theta = a$.

**Sol.** The equation of the curve is $r\theta = a$.

Put $\qquad r = \dfrac{1}{u} \quad \therefore \quad \dfrac{\theta}{u} = a \quad \therefore \quad u = \dfrac{\theta}{a}$.

As $\qquad u \to 0,\ \dfrac{\theta}{a} \to 0 \quad$ *i.e.*, $\quad \theta \to 0 \quad \therefore \quad \theta_1 = 0$

Since $\qquad u = \dfrac{\theta}{a}$, therefore $\dfrac{du}{d\theta} = \dfrac{1}{a}$

$$p = \underset{\theta \to \theta_1}{\text{Lt}}\left(-\frac{d\theta}{du}\right) = \underset{\theta \to 0}{\text{Lt}} -a = -a.$$

$\therefore$ Equation of the asymptote is

$$p = r\sin(\theta_1 - \theta) \quad \text{or} \quad -a = r\sin(-\theta) = -r\sin\theta$$

*i.e.*, $$r\sin\theta = a.$$

**Remark.** The student should remember the following results from Trigonometry.

**1.** If $\sin\theta = 0$ ; then $\theta = n\pi$ where $n$ is any integer.

**2.** If $\cos\theta = 0$, then $\theta = (2n + 1)\dfrac{\pi}{2}$.

**3.** $\sin(n\pi + \theta) = (-1)^n \sin\theta$.

**4.** $\cos(n\pi + \theta) = (-1)^n \cos\theta$.

**5.** If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$.

**6.** If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$.

**7.** If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$.

**8.** $\dfrac{1}{(-1)^n} = (-1)^n$.

**NOTES**

**Example 9.** *Find the asymptotes of the curve r cos θ = a sin² θ.*

**Sol.** The equation of the curve is $r \cos \theta = a \sin^2 \theta$

Put $\quad r = \dfrac{1}{u} \quad \therefore \quad \dfrac{1}{u} \cos \theta = a \sin^2 \theta$

$$\therefore \qquad u = \frac{\cos \theta}{a \sin^2 \theta}$$

As $\quad u \to 0, \; \dfrac{\cos \theta}{a \sin^2 \theta} \to 0, \quad i.e., \quad \cos \theta \to 0$

$$\therefore \qquad \theta \to (2n + 1)\frac{\pi}{2} \; i.e., \; \theta_1 = (2n + 1)\frac{\pi}{2}$$

Since $\quad u = \dfrac{\cos \theta}{a \sin^2 \theta}$

$$\therefore \qquad \frac{du}{d\theta} = \frac{1}{a}\left[\frac{\sin^2 \theta (- \sin \theta) - \cos \theta \,.\, 2 \sin \theta \cos \theta}{\sin^4 \theta}\right]$$

$$= \frac{(- \sin^3 \theta - 2 \sin \theta \cos^2 \theta)}{a \sin^4 \theta}$$

$$= \frac{- \sin \theta (\sin^2 \theta + 2 \cos^2 \theta)}{a \sin^4 \theta} = \frac{-(\sin^2 \theta + 2 \cos^2 \theta)}{a \sin^3 \theta}$$

$$p = \underset{\theta \to \theta_1}{\text{Lt}} \left(-\frac{d\theta}{du}\right) = \underset{\theta \to (2n+1)\pi/2}{\text{Lt}} \frac{a \sin^3 \theta}{\sin^2 \theta + 2 \cos^2 \theta}$$

or $$p = a\,\frac{((-1)^n)^3}{((-1)^n)^2 + 0}$$

$$\left[\begin{array}{l} \because \quad \sin (2n+1)\dfrac{\pi}{2} = \sin \left(n\pi + \dfrac{\pi}{2}\right) = (-1)^n \sin \dfrac{\pi}{2} = (-1)^n \\ \text{and } \cos(2n+1)\dfrac{\pi}{2} = \cos\left(n\pi + \dfrac{\pi}{2}\right) = (-1)^n \cos \dfrac{\pi}{2} = 0 \end{array}\right]$$

or $$p = a(-1)^n$$

∴ Equation of the asymptote is

$$p = r \sin (\theta_1 - \theta)$$

or $$a(-1)^n = r \sin \left[(2n+1)\frac{\pi}{2} - \theta\right]$$

or $$a(-1)^n = r \sin \left(n\pi + \frac{\pi}{2} - \theta\right)$$

or $$a(-1)^n = (-1)^n r \sin \left(\frac{\pi}{2} - \theta\right) \quad \text{or} \quad a = r \cos \theta.$$

**Example 10.** *Find the asymptotes of the curve r = a sec θ + b tan θ.*

**Sol.** The equation of the curve is $r = a \sec \theta + b \tan \theta$.

Put $\quad r = \dfrac{1}{u}$ and change all *t*-ratios in sin θ, cos θ.

$$\therefore \qquad \frac{1}{u} = \frac{a}{\cos \theta} + \frac{b \sin \theta}{\cos \theta} = \frac{a + b \sin \theta}{\cos \theta}$$

$$\therefore \qquad u = \frac{\cos \theta}{a + b \sin \theta}$$

As $\quad u \to 0, \; \dfrac{\cos \theta}{a + b \sin \theta} \to 0 \quad i.e., \quad \cos \theta \to 0$

$\therefore \qquad \theta \to (2n+1)\dfrac{\pi}{2} \qquad$ *i.e.*, $\quad \theta_1 = (2n+1)\dfrac{\pi}{2}$

Since $\qquad u = \dfrac{\cos\theta}{a + b\sin\theta}$

$$\therefore \qquad \frac{du}{d\theta} = \frac{(a + b\sin\theta)(-\sin\theta) - \cos\theta\,(b\cos\theta)}{(a + b\sin\theta)^2}$$

or
$$\frac{du}{d\theta} = -\frac{(a\sin\theta + b)}{(a + b\sin\theta)^2}$$

$$p = \underset{\theta \to \theta_1}{\text{Lt}} \left(-\frac{d\theta}{du}\right) = \underset{\theta \to (2n+1)\pi/2}{\text{Lt}} \frac{(a + b\sin\theta)^2}{(a\sin\theta + b)}$$

$$= \frac{\left[a + b\sin(2n+1)\dfrac{\pi}{2}\right]^2}{\left[a\sin(2n+1)\dfrac{\pi}{2} + b\right]} = \frac{[a + b(-1)^n]^2}{[a(-1)^n + b]}$$

$$\because \quad \sin(2n+1)\frac{\pi}{2} = \sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n \sin\frac{\pi}{2} = (-1)^n .$$

$\therefore$ Equation of asymptotes is $p = r\sin(\theta_1 - \theta)$

or
$$p = r\sin\left((2n+1)\frac{\pi}{2} - \theta\right) = r\sin\left(n\pi + \frac{\pi}{2} - \theta\right) = r(-1)^n \sin\left(\frac{\pi}{2} - \theta\right)$$

or
$$p = r(-1)^n\cos\theta \quad \text{or} \quad \frac{[a + b(-1)^n]^2}{[a(-1)^n + b]} = (-1)^n\, r\cos\theta.$$

**Note.** $\because$ If $n$ is *even* ; then $(-1)^n = 1$.

$\therefore$ Equation of asymptote is $\dfrac{(a+b)^2}{a+b} = r\cos\theta \quad$ *i.e.*, $\quad r\cos\theta = a + b$

If $n$ is *odd* ; then $(-1)^n = -1$

$\therefore$ Equation of asymptote is $\dfrac{(a-b)^2}{(-a+b)} = -r\cos\theta$

or
$$\frac{(a-b)^2}{a-b} = r\cos\theta \quad \textit{i.e.,} \quad r\cos\theta = (a - b).$$

## EXERCISE D

*Find the asymptotes of the following curves :*

**1.** (*i*) $r = \dfrac{a\theta}{\theta - 1}$ $\qquad$ (*ii*) $r^2\theta = a^2$

(*iii*) $r(\theta^2 - \pi^2) = 2a\theta$.

**2.** (*i*) $r = \dfrac{2a}{1 - 2\cos\theta}$ $\qquad$ (*ii*) $r = \dfrac{2a}{1 + 2\cos\theta}$

(*iii*) $r\cos\theta = a\sin\theta$ or $r = a\tan\theta$.

NOTES

3. (*i*) $r \sin n\theta = a$ (*ii*) $r \sin 2\theta = a$.
4. (*i*) $r \cos 2\theta = a \sin 3\theta$ (*ii*) $r \sin \theta = 2 \cos 2\theta$.
   (*iii*) $r \cos \theta = a \cos 2\theta$.
5. (*i*) $r^n \sin n\theta = a^n$ or $r^n = a^n \operatorname{cosec} n\theta$ where $n$ is a positive integer > 1.
   (*ii*) $r = \dfrac{3a \sin \theta \cos \theta}{\sin^3 \theta + \cos^3 \theta}$.
   [**Hint.** $u = 0$ gives $\sin^3 \theta = -\cos^3 \theta$ *i.e.*, $\tan \theta = -1$.]
6. (*i*) $r = 4(\sec \theta + \tan \theta)$ (*ii*) $r = a \operatorname{cosec} \theta + b$
   (*iii*) $r^2 = a^2 (\sec^2 \theta + \operatorname{cosec}^2 \theta)$ (*iv*) $r = a + b \cot n\theta$
   (*v*) $r = a \operatorname{cosec} \theta + b \cot \theta$.
7. $r(1 - e^{\theta}) = a$. **8.** $r \log \theta = a$.
9. (*i*) $r(\pi + \theta) = ae^{\theta}$ (*ii*) $r\theta \cos \theta = a \cos 2\theta$.

## Answers

1. (*i*) $r \sin (\theta - 1) = a$ (*ii*) $\theta = 0$ (*iii*) $r \sin \theta = -a$.
2. (*i*) $r \sin \left(\dfrac{\pi}{3} \pm \theta\right) = \dfrac{-2a}{\sqrt{3}}$ (*ii*) $r \sin \left(\dfrac{2\pi}{3} \pm \theta\right) = \dfrac{2a}{\sqrt{3}}$ (*iii*) $r \cos \theta = (-1)^n a$.
3. (*i*) $r \sin \left(\theta - \dfrac{m\pi}{n}\right) = a\dfrac{(-1)^m}{n}$ where $m$ is any integer
   (*ii*) $-a\dfrac{(-1)^n}{2} = r \sin \left(\dfrac{n\pi}{2} - \theta\right)$
4. (*i*) $\dfrac{a}{2} \sin \left(\dfrac{n\pi}{2} + \dfrac{3\pi}{4}\right) = r \sin \left[(2n+1)\dfrac{\pi}{4} - \theta\right]$
   (*ii*) $r \sin \theta = 2$ (*iii*) $r \cos \theta + a = 0$.
5. (*i*) $\theta = \dfrac{m\pi}{n}$, where $m$ is any integer (*ii*) $\sqrt{2}\, r \sin \left(\dfrac{\pi}{4} + \theta\right) + a = 0$.
6. (*i*) $4[1 + (-1)^n] = (-1)^n r \cos \theta$ (*ii*) $r \sin \theta = a$
   (*iii*) $\pm a(-1)^n = r \sin \left(\dfrac{n\pi}{2} - \theta\right)$
   (*iv*) $r \sin \left(\theta - \dfrac{m\pi}{n}\right) = \dfrac{b}{n}$ where $m$ is an integer.
   (*v*) $a(-1)^n + b = (-1)^n r \sin \theta$.
7. $r \sin \theta + a = 0$. **8.** $a = r \sin (\theta - 1)$.
9. (*i*) $r \sin \theta\, e^{\pi} + a = 0$
   (*ii*) $r \sin \theta = a$, $(2n + 1)\pi r \cos \theta + 2a = 0$, where $n$ is any *integer*.

# 12. SINGULAR POINTS

NOTES

**STRUCTURE**

Singular Point
Concavity and Convexity
Point of Inflexion
Criteria for concavity, Convexity and point of inflexion
Concavity and Convexity for Polar Curves with Respect to the pole
Multiple Points
Classification of Double Points
Tangents at the Origin
Working Rule for Finding The Nature of Origin Which is a Double Point.
Another Method of Finding The Position of Double Points
Kinds Of Cusps
Working Rule to Find The Nature of Cusp at the Origin

## SINGULAR POINT

**Def.** *A point on the curve at which the curve behaves in an extra ordinary manner is called a* **singular point.**

There are two types of singular points:

(*i*) *Points of inflexion.*
(*ii*) *Multiple points.*

## CONCAVITY AND CONVEXITY

**Def.** Let P be a point on the given curve $y = f(x)$, such that the tangent of P is not parallel to $y$-axis.

NOTES

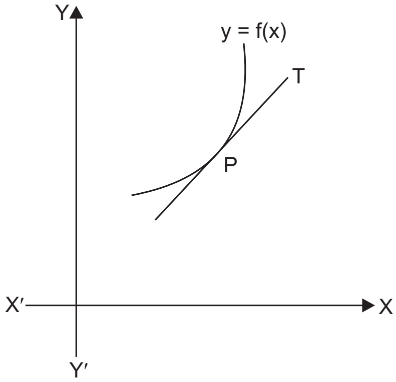



Fig. 1

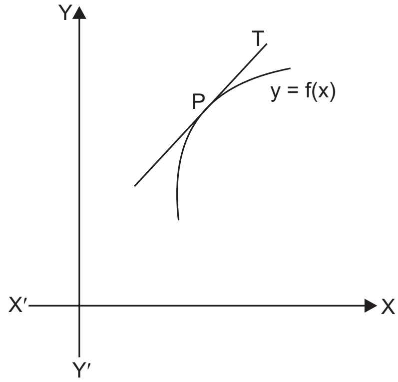



Fig. 2

(*i*) **Concave upwards (or convex downwards)** *at P if in the neighbourhood of* P, *the curve lives P, the curve lies above the tangent at P on both sides.* [See Fig. (1)], and

(*ii*) **Concave downward (or convex upwards)** *at P if in the neighbourhood of P, the curve lies below the tangent at P on both sides.*

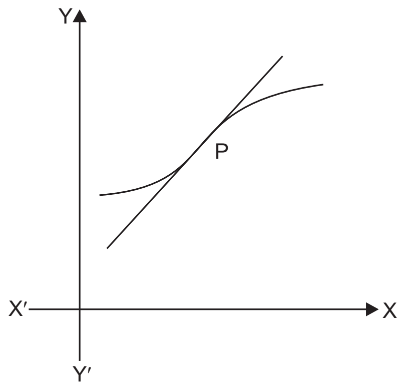
## POINT OF INFLEXION

**Def.** *A point on the curve at which the curve changes from concavity to convexity or vice-versa is called a* **point of inflexion.**

Since the change from the concavity to convexity of *vice versa* is possible only if the curve crosses the tangent at a point.

$\therefore$ the point of inflexion may also be defined *as a point on the curve at which the curve crosses the tangent.*

**Note.** A point of inflexion is a *singular point* (*i.e.*, an unusual point) on the curve, for the tangent does not usuaully cross the curve, as it does at the point of inflexion.




## CRITERIA FOR CONCAVITY, CONVEXITY AND POINT OF INFLEXION

**If y = f(x) be a curve, then prove that**

(*i*) **the curve is concave upwards at a point P on it if** $\frac{d^2y}{dx^2}$ **is positive.**

(*ii*) **the curve is convex upwards at a point P on it if** $\frac{d^2y}{dx^2}$ **is negative.**

(*G.N.D.U. 1981*)

**NOTES**

(*iii*) **the curve has a point of inflexion at P if**

(*a*) $\mathbf{\frac{d^2y}{dx^2} = 0}$ **and**

(*b*) $\mathbf{\frac{d^2y}{dx^2}}$ **changes sign as x passes through P i.e.,** $\mathbf{\frac{d^3y}{dx^3} \neq 0}$**.**

**Proof.** Let P($x$, $y$) be any point on the curve $y = f(x)$. Take a neighbouring point Q($x + h$, $y + k$) on both sides of P. From Q draw QN ⊥ OX, meeting the tangent at P to the curve in R.

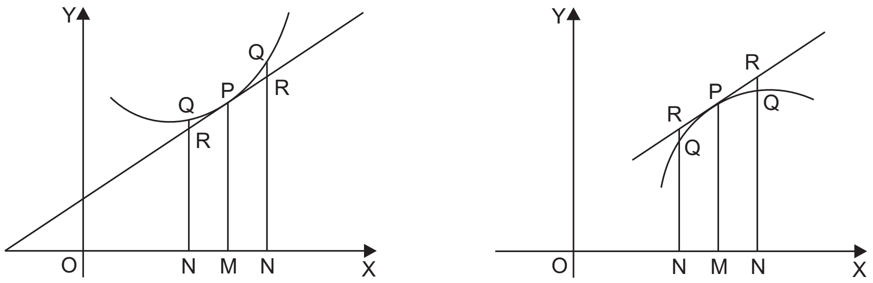



**Fig. (*i*)** **Fig. (*ii*)**

The equation of the tangent at P($x$, $y$) is

$$Y - y = f'(x)(X - x)$$

or $$Y = y + f'(X - x).$$

It meets QN, where X = $x + h$, in the point, so that

$$RN = Y = y + f'(x)[(x + h) - x]$$
$$= y + hf'(x) = f(x) + hf'(x)$$

Also QN = ordinate of Q *i.e.*, the ordinate cooresponding to the abscissa $x + h$ $= f(x + h)$.

Now $$QN - RN = f(x + h) - f(x) - hf'(x) \qquad \text{...(A)}$$

[*Expand f(x + h) by Taylor's Theorem with remainder after two terms*]

$$= \left[f(x) + hf'(x) + \frac{h^2}{2!}f''(x + \theta h)\right] - f(x) - hf'(x)$$

where $0 < \theta < 1$

$$= \frac{h^2}{2!}f''(x + \theta h). \qquad \text{...(1)}$$

If $f''(x)$ is continuous and non-zero, and since $h$ is very small, then $f''(x + \theta h)$ has the same sign as $f''(x)$, whatever be the sign of $h$.

Thus from (1), it follows that the sign of QN – RN depends on $f''(x)$.

(*i*) The curve will be **concave upwards** (or convex downwards) at P [as in fig. (*i*)] if QN > RN (for both $h$ + ve and – ve) *i.e.* if QN – RN is + ve *i.e.* if $f''(x)$ is + ve or if $\mathbf{\frac{d^2y}{dx^2}}$ **is positive.**

(*ii*) The curve will be **convex upwards** (or *concave downwards*) at P[as in fig. (*ii*) if QN < RN (for both $h$ + ve and – ve), *i.e.*, if QN – RN is – ve *i.e.*, if $f''(x)$ is – ve or if $\frac{d^2y}{dx^2}$ **is negative.**

**NOTES**

(*iii*) **For the point of inflexion:**

Let $f''(x) = 0$ and $f'''(x) \neq 0$.

Expanding $f(x + h)$ by Taylor's Theorem with remainder after three terms in (A), we get

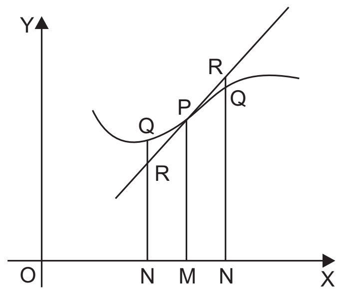
$$QN - RN = \left[f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x + \theta_1 h)\right] - f(x) - hf'(x), \text{ where } 0 < \theta_1 < 1$$

$$= \frac{h^3}{3!}f'''(x + \theta_1 h) \qquad \ldots(2) \qquad | \; f''(x) = 0 \; (given)$$

$\because$ $f'''(x)$ is a continuous function of $x$ at P and $f'''(x) = 0$

$\therefore$ $f'''(x + \theta_1 h)$ has the same sign as $f'''(x)$ in the neighbourhood of P.

$\therefore$ There is a **point of inflexion at P if f ″(x) = 0 and $f'''(x) \neq 0$.**




Thus we have

*(i) A curve is concave upwards if* $\frac{d^2y}{dx^2}$ *is + ve.*

*(ii) A curve is convex upwards if* $\frac{d^2y}{dx^2}$ *is – ve.*

*and (iii) At the point of inflexion,* $\frac{d^2y}{dx^2} = 0$ *and* $\frac{d^3y}{dx^3} = 0$

} **Remember**

**Cor. 1.** The above result can be generalised.

Thus if $f''(x) = f'''(x) \ldots\ldots = f^{n-1}(x) = 0$ and $f^n(x) \neq 0$, then

(*i*) **the curve has a point of inflexion at P if n is odd**

(*ii*) **and the curve is concave upwards or convex upwards according as $f^n(x) > 0$ or < 0, and n is even.**

**Note.** In the above investigation, we have assumed that $\frac{dy}{dx}$ is finite. **If $\frac{dy}{dx}$ becomes infinite, then we must find the points of inflexion by considering $\frac{d^2y}{dx^2}$.**

## SOLVED EXAMPLES

**Example 1.** (*a*) *Prove that $y = e^x$ is everwhere concave upwards.*

(*b*) *Prove that the curve $y = \log x$ is convex upwards everywhere.*

NOTES

**Solution.** (*a*) The equation the curve is $y = e^x$.

$$\therefore \qquad \frac{dy}{dx} = e^x \quad \text{and} \quad \frac{d^2y}{dx^2} = e^x$$

$\because$ $\frac{d^2y}{dx^2}$ is always + ve. $\therefore$ the curve is concave upwards.

(*b*) The equation of the curve is $y = \log x$ $(x > 0)$.

$$\therefore \qquad \frac{dy}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{-1}{x^2}$$

$$\because \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2} \text{ is } - \text{ve for all values of } x.$$

$\therefore$ The curve is convex upwards everywhere.

**Example 2.** *Find the range of values of x for which the following curves are convaves upwards or downwards:*

(*a*) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ (*b*) $y = 3x^5 - 40x^3 + 3x - 20$

(*c*) $y = (x^2 + 4x + 5)\, e^{-x}$

*Find also the points of inflexion in each case.*

**Solution.** (*a*) The curve is $y = x^4 - 6x^3 + 12x^2 + 5x + 7$ ...(1)

$$\therefore \qquad \frac{dy}{dx} = 4x^3 - 18x^2 + 24x + 5$$

$$\frac{d^2y}{dx^2} = 12x^2 - 36x + 24 = 12(x^2 - 3x + 2) = 12(x-1)(x-2).$$

Now in $(-\infty, 1)$, $\frac{d^2y}{dx^2} > 0$ $\therefore$ curve is concave upwards.

In $(1, 2)$, $\frac{d^2y}{dx^2} < 0$ $\therefore$ curve is concave downwards.

In $(2, \infty)$, $\frac{d^2y}{dx^2} > 0$, $\therefore$ curve is concave upwards.

Putting $\frac{d^2y}{dx^2} = 0$, $x = 1, 2$.

Also $\frac{d^3y}{dx^3} = 24x - 36$

at $x = 1$, $\frac{d^3y}{dx^3} = 24 - 36 \neq 0$

at $x = 2$, $\frac{d^3y}{dx^3} = 48 - 36 \neq 0$ $\therefore$ There are pts. of inflexion at $x = 1, 2$.

From (1), when $x = 1$, $y = 19$ and when $x = 2$, $y = 33$.

Hence the points of inflextion are (1, 19) and (2, 33).

**Note.** We can also proceed like this:

where $x$ is slightly $< 1$, $\frac{d^2y}{dx^2} > 0$; when $x$ is slight $> 1$, $\frac{d^2y}{dx^2} < 0$ $\because$ $\frac{d^2y}{dx^2}$ changes sign from + ve to – ve at $x = 1$ $\therefore$ there is a point of inflexion at $x = 1$, Similarly for $x = 2$.

(*b*) The equation of the curve is $y = 3x^5 - 40x^3 + 3x - 20$ ...(1)

$$\therefore \quad \frac{dy}{dx} = 15x^4 - 120x^2 + 3$$

NOTES

and $$\frac{d^2y}{dx^2} = 60x^3 - 240x = 60x\,(x - 2)\,(x + 2)$$

In $(-\infty, -2)$, $\frac{d^2y}{dx^2}$ is – ve, $\therefore$ curve is concave downwards.

In $(-2, 0)$, $\frac{d^2y}{dx^2}$ is + ve, $\therefore$ curve is concave upwards.

In $(0, 2)$, $\frac{d^2y}{dx^2}$ is – ve, $\therefore$ curve is concave downwards.

and in $(2, \infty)$, $\frac{d^2y}{dx^2}$ is + ve, $\therefore$ curve is concave upwards.

Putting $\frac{d^2y}{dx^2} = 0$, $x = 0 \pm 2$.

Also $\frac{d^3y}{dx^3} = 180x^2 - 240$ which does not vanish for $x = 0, \pm 2$

$\therefore$ there are points of inflection at $x = 0, \pm 2$.
From (1), when $x = 0$, $y = -20$;
When $x = -2$, $y = 198$ and when $x = 2$, $y = -238$.
Hence the points of inflexion are

$$(0, -20), (-2, 198) \text{ and } (2, -238).$$

(*c*) The equation of the curve is $y = (x^2 + 4x + 5)e^{-x}$ ...(1)

$$\therefore \quad \frac{dy}{dx} = (x^2 + 4x + 5)\,.\,e^{-x}(-1) + e^{x}(2x + 4)$$
$$= (-x^2 - 4x - 5 + 2x + 4)e^{-x} = -(x^2 + 2x + 1)e^{-x}$$

and $$\frac{d^2y}{dx^2} = -(x^2 + 2x + 1)e^{-x}(-1) - (2x + 2)e^{-x}$$
$$= (x^2 + 2x + 1 - 2x - 2)e^{-x} = (x^2 - 1)e^{-x}$$
$$= (x + 1)\,(x - 1)e^{-x}.$$

In $(-\infty, -1)$, $\frac{d^2y}{dx^2} > 0$ $\therefore$ curve is concave upwards.

In $(-1, 1)$, $\frac{d^2y}{dx^2} < 0$ $\therefore$ curve is concave downwards.

and in $(1, \infty)$, $\frac{d^2y}{dx^2} > 0$ $\therefore$ curve is concave upwards.

Putting $\frac{d^2y}{dx^2} = 0$, $x = \pm 1$.

$\because$ $\frac{d^2y}{dx^2}$ changes sign at $x = +1$ and $-1$.

$\therefore$ there are points of inflexion at $x = \pm 1$.

When $x = -1$, from (1), $y = (1 - 4 + 5)e^1 = 2e$

and when $x = 1$, from (1), $y = (1 + 4 + 5)e^{-1} = 10e^{-1} = \dfrac{10}{e}$.

Hence the points of inflexion are $(-1, 2e)$ and $\left(1, \dfrac{10}{e}\right)$.

**Example 3.** *Find the value of x for which the curve*

$$54y = (x + 5)^2 (x^3 - 10)$$

*has a point of inflexion.*

**Solution.** Equation of the curve is

$$5y = (x + 5)^2 (x^3 - 10)$$

$$\therefore \quad 54\frac{dy}{dx} = (x + 5)^2 (3x^2) + 2(x + 5)(x^3 - 10)$$

$$= (x + 5) [3x^2 (x + 5) + 2(x^3 - 10)]$$

$$= (x + 5) [3x^3 + 15x^2 + 2x^3 - 20]$$

$$= (x + 5) [5x^3 + 15x^2 - 20]$$

$$= 5(x + 5) (x^3 + 3x^2 - 4)$$

or $$\frac{54}{5} \cdot \frac{d^2y}{dx^2} = (x + 5) (3x^2 + 6x) + (1) (x^3 + 3x^2 - 4)$$

$$= 3x^3 + 21x^2 + 30x + x^3 + 3x^2 - 4$$

$$= 4x^3 + 24x^2 + 30x - 4$$

$$\frac{d^3y}{dx^3} = \frac{54}{5} [12x^2 + 48x + 30] = \frac{5}{9} [2x^2 + 8x + 5]$$

For points of inflexion, $\dfrac{d^2y}{dx^2} = 0$

or $4x^3 + 24x^2 + 30x - 4 = 0$ or $2x^3 + 12x^2 + 15x - 2 = 0$

or $(x + 2) (2x^2 + 8x - 1) = 0 \quad \therefore \quad x = -2, \dfrac{-8 \pm \sqrt{64 + 8}}{4}$

$$\therefore \quad x = -2 \quad \text{or} \quad \frac{-4 \pm 3\sqrt{2}}{2}$$

When $x = -2$, $\dfrac{d^3y}{dx^3} = \dfrac{5}{9} [8 - 16 + 5] \neq 0$

When $x = \dfrac{1}{2} [-4 \pm 3\sqrt{2}]$, $\dfrac{d^3y}{dx^3} \neq 0$

Hence points of inflexion are given by

$$x = -2, \frac{1}{2}(-4 \pm 3\sqrt{2}).$$

**Example 4.** *Show that origin is a point of inflexion of the curve $a^{m-1} \,.\, y = x^m$ if $m$ is odd and greater than 2.*

**Solution.** The equation of the curve is $a^{m-1} \,.\, y = x^m$.

NOTES

or $$y = \frac{x^m}{a^{m-1}}$$

$\therefore$ $$\frac{dy}{dx} = \frac{m}{a^{m-1}} \,.\, x^{m-1};\quad \frac{d^2y}{dx^2} = \frac{m(m-1)}{a^{m-1}} \,.\, x^{m-2}$$

$$\frac{d^3y}{dx^3} = \frac{m(m-1)(m-2)}{a^{m-1}} \,.\, x^{m-3};\ \ldots\ldots\ \frac{d^my}{dx^m} = \frac{m\,!}{a^{m-1}}$$

For points of inflexion, $\dfrac{d^2y}{dx^2} = 0$

or $$\frac{m(m-1)}{a^{m-1}} \,.\, x^{m-2} = 0 \quad \text{or} \quad x^{m-2} = 0$$

or $$x = 0,\ \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = \ldots\ldots = \frac{d^{m-1}y}{dx^{m-1}} = 0 \text{ and } \frac{d^my}{dx^m} \neq 0.$$

Thus there is a point of inflexion at $x = 0$ if $m$ is odd and no point of inflexion if $m$ is even.

Hecne, the origin is a point of inflexion if $m$ is odd and > 2.

**Example 5.** *Find the points of inflexion on the curve*

$$x = a(2\theta - \sin\theta), \quad y = a(2 - \cos\theta).$$

**Solution.** Here $\dfrac{dx}{d\theta} = a(2 - \cos\theta)$ and $\dfrac{dy}{d\theta} = a\sin\theta$

$\therefore$ $$\frac{dy}{dx} = \frac{\dfrac{dx}{d\theta}}{\dfrac{dy}{d\theta}} = \frac{\sin\theta}{2 - \cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{2 - \cos\theta}\right) . \frac{d\theta}{dx}$$

$$= \frac{(2 - \cos\theta)\cos\theta - \sin\theta\,(\sin\theta)}{(2 - \sin\theta)^2} \times \frac{1}{a(2 - \cos\theta)}$$

$$= \frac{2\cos\theta\,(\cos^2\theta + \sin^2\theta)}{a(2 - \sin\theta)^3} = \frac{2\cos\theta - 1}{a(2 - \cos\theta)^3}$$

For points of inflexion, $\dfrac{d^2y}{dx^2} = 0$ or $\dfrac{2\cos\theta - 1}{a(2 - \cos\theta)^3} = 0$

which gives $$\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3} \quad \therefore \quad \theta = 2n\pi \pm \frac{\pi}{3}.$$

where $n$ is any integer.

When $\theta$ passes through each of these values, $\dfrac{d^2y}{dx^2}$ changes sign. Hence there are points of inflexion corresponding to every value of $\theta$ given above.

NOTES

The co-ordinates of the points of inflexion are

$$\left[a\left(4n\pi \pm \frac{2\pi}{3} \pm \frac{\sqrt{3}}{2}\right), \frac{3a}{2}\right]$$

$$\because \quad \sin\left(2n\pi \pm \frac{\pi}{3}\right) = \sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2}$$

## EXERCISE – 1

**1.** Discuss the concavity and convexity of the curve

$$y = (\sin x + \cos x)e^x$$

when $0 \le x \le 2\pi$.

Find also the points of inflexion.

**2.** Find points of inflexion of the following curves:

(*a*) $xy = a^2 \log\left(\frac{y}{a}\right)$ (*b*) $x = (\log y)^3$.

**3.** Show that the line joining the points of inflexion of the curve $y^2(x - a) = x^2(x + a)$ substends an angle of $\pi/3$ at the origin.

**4.** Show that points of inflexion of the curve

$$y^2 = (x - a)^2 (x - b)$$

lie on the line $3x + a = 4b$.

**5.** For the curve $y = x^3 + bx^2 + c$, where $b < 0$ show that the point of inflexion is equidistant from the maximum and minimum points.

**6.** Show that abscissae of the points of inflexion on the curve $y^2 = f(x)$ satisfy the equation $[f'(x)]^2 = 2f(x) \,.\, f''(x)$.

### Answers

**1.** The points of inflexion are $\left(\frac{\pi}{4}, \sqrt{2}e^{\pi/4}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}e^{5\pi/4}\right)$.

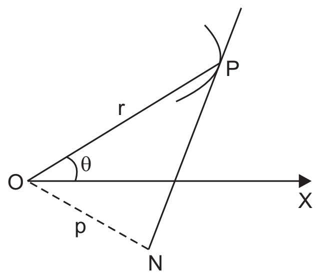
**2.** (*a*) The point of inflexion is $\left(\frac{3}{2}ae^{-3/2}, ae^{3/2}\right)$

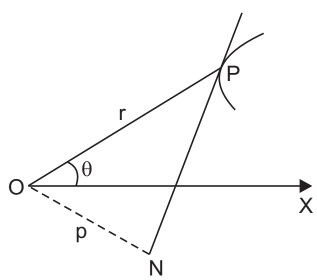
(*b*) The points of inflexion are (0, 1) and $(8, e^2)$.

## CONCAVITY AND CONVEXITY FOR POLAR CURVES WITH RESPECT TO THE POLE

Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$.

Draw ON $\perp$ on the tangent at P and let ON = $p$.



NOTES

Then the curve is said to be concave or convex with respect to O according as $p$ increases or decreases with the increase in $r$.

*i.e.*, according as $\frac{dp}{dr}$ is + ve or – ve.

Now we know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Different both sides w.r.t. to $r$, we have

$$-\frac{2}{p^2}\cdot\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\cdot 2\left(\frac{dr}{d\theta}\right)\cdot\frac{d^2r}{d\theta^2}\cdot\frac{d\theta}{dr}$$

or

$$\frac{1}{p^3}\cdot\frac{dp}{dr} = \frac{1}{r^3} + \frac{2}{r^5}\cdot\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^4}\cdot\frac{d^2r}{d\theta^2}$$

$$\therefore \qquad \frac{dp}{dr} = \frac{p^3}{r^5}\left[r^2 + 2\cdot\left(\frac{dr}{d\theta}\right)^2 - r\cdot\frac{d^2r}{d\theta^2}\right]$$

Now sign of $\frac{dp}{dr}$ is the same as that of the expression

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\cdot\frac{d^2r}{d\theta^2}$$

Hence the curve is concave or convex at P($r$, θ) according as $\mathbf{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\,.\,\frac{d^2r}{d\theta^2}}$ **is positive or negative.**

**Cor. Condition for point of inflexion.**

There is a point of inflexion at P, if

(*i*) $\mathbf{r^2 + 2\,.\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = 0}$ **or** $\mathbf{u + \frac{d^2u}{d\theta^2} = 0}$, **where** $\mathbf{u = \frac{1}{r}}$

(*ii*) **the experssion** $\mathbf{u + \frac{d^2u}{d\theta^2}}$ **change sign in passing through P.**

## SOLVED EXAMPLES

**Example 6.** *Determine whether the spiral r cosh θ = a is concave or convex towards the pole.*

**Solution.** Given curve is $r \cosh\theta = a$ or $r = a\,\text{sech}\,\theta$

$$\frac{dr}{d\theta} = -a\,\text{sech}\,\theta\,.\,\tanh\theta$$

$$\frac{d^2r}{d\theta^2} = -a[\text{sech}\,\theta\,.\,\text{sech}^2\,\theta - \tanh\theta\,.\,\text{sech}\,\theta\,.\,\tanh\theta]$$

$$= -a[\text{sech}^3\,\theta - \tanh^2\theta\,.\,\text{sech}\,\theta]$$

$$= -a\,\text{sech}\,\theta\,.\,[\text{sech}^2\,\theta - \tanh^2\theta]$$

NOTES

$\therefore \quad r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}$

$= a^2 \text{ sech}^2 \theta + 2a^2 \text{ sech}^2 \theta \cdot \tanh^2 \theta + a^2 \text{ sech}^2 \theta \cdot (\text{sech}^2 \theta - \tanh^2 \theta)$

$= a^2 \text{ sech}^2 \theta + 2a^2 \text{ sech}^2 \theta \cdot \tanh^2 \theta + a^2 \text{ sech}^4 \theta - a^2 \text{ sech}^2 \theta \tanh^2 \theta$

$= a^2 \text{ sech}^2 \theta \cdot [1 + \tanh^2 \theta + \text{sech}^2 \theta] = 2a^2 \cdot \text{sech}^2 \theta = +\text{ve}.$

$| \because \quad \text{sech}^2 \theta + \tanh^2 \theta = 1$

Hence the curve is *concave* towards pole.

**Example 7.** *Show that the curve* $r\sqrt{\theta} = a$ *has a point of inflexion at a distance of* $\sqrt{2}a$ *from the pole.*

**Solution.** Given curve is $r\sqrt{\theta} = a$ ...(*i*)

Putting $r = \frac{1}{u}$, we get $u = \frac{\sqrt{\theta}}{a}$

$$\therefore \quad u_1 = \frac{1}{a} \cdot \frac{1}{2\sqrt{\theta}}$$

$$u_2 = \frac{1}{2a}\left(-\frac{1}{2}\right) \cdot \theta^{-3/2} = -\frac{1}{4a\theta\sqrt{\theta}}$$

For points of inflexion, $u + u_2 = 0$ *i.e.* $\frac{\sqrt{\theta}}{a} - \frac{1}{4a\,\theta\sqrt{\theta}} = 0$

$$4\theta^2 - 1 = 0 \quad \text{or} \quad \theta^2 = \frac{1}{4} \quad \text{or} \quad \theta = \pm\frac{1}{2}$$

When $\theta = \frac{1}{2}$, $r = 2\sqrt{a}$

and when $\theta = -\frac{1}{2}$, $r$ is imaginary.

Hence point of inflexion is at $r = \sqrt{2}a$ .

**Example 8.** *Show that the points of inflexion on the curve* $r = a\theta^n$ *are given by* $r = a[-n(n+1)]^{n/2}$.

**Solution.** The equation of the given curve is $r = a\theta^n$ ...(1)

$$\therefore \quad \frac{dr}{d\theta} = na\, q^{n-1} \quad \text{and} \quad \frac{d^2r}{d\theta^2} = n(n-1)\, a\, q^{n-2}.$$

For the points of inflexion, $r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = 0$

or $\quad a^2\theta^{2n} + 2n^2a^2\, \theta^{2n-2} - a\theta^n \cdot n(n-1)\, a\theta^{n-2} = 0$

or $\quad a^2\theta^{2n-2}[\theta^2 + 2n^2 - n(n-1)] = 0$

or $\quad a^2\theta^{2n-2}[\theta^2 + n^2 + n] = 0.$

$\therefore$ Either $\theta = 0$ or $\theta^2 + n^2 + n = 0$

When $\theta = 0$, all the derivatives will become zero. $\therefore$ $\theta = 0$ cannot give any point of inflexion.

Thus, $\theta^2 + n^2 + n = 0$ or $\theta^2 = -n(n+1)$ $\therefore$ $\theta = [-n(n+1)]^{1/2}$

$\therefore$ From (1), $r = a\theta^n = a[-n(n+1)]^{n/2}$

Hence the points of inflexion are given by $r = a[-n(n+1)]^{n/2}$.

## MULTIPLE POINTS

NOTES

**Def.:** *A point on the curve through which more than one branches of the curve pass is called a* **multiple point.**

### Double Point (Def.)

*A point on a curve is called a* **double point** *if two branches of the curve pass through it.*

There are in general, two tangents at a double point which may be real and distinct or real and coincident or imaginary.

### Triple Point (Def.)

*A point through which three branches of the curve pass is called a* **triple point.**

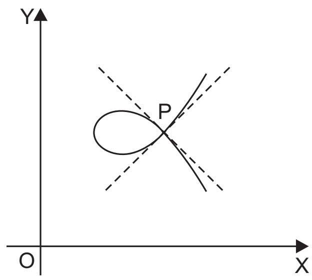
### Multiple Point of rth Order (Def.)

*If through a point on the curve, r branches of the curve pass, then that point is called a* **multiple point of rth order.**

In general, $r$ tangents (real and distinct, coincident or imaginary) can be drawn through a multiple point of order $r$.

## CLASSIFICATION OF DOUBLE POINTS

There are three kinds of double points.

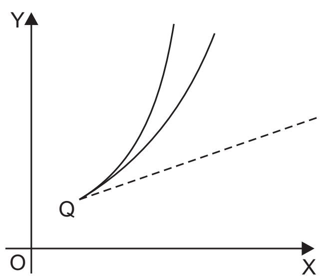


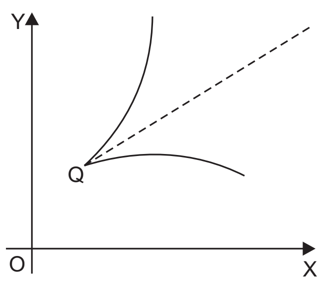

### Node

**Def.** A node is a point on the curve through which pass *two real branches of the curve*, and the two tangents at which are *real and distinct*. Thus P is a node.

### Cusp

**Def.** A double point on the curve through which two real branches of the curves pass and the tangents at which are *real and* coincident is called a cusp.



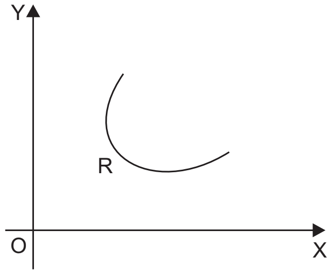




Thus Q is a cusp.

NOTES

## Conjugate Point or Isolated Point

**Def.** A conjugate point or an isolated point on a curve is a point in the neighbourhood of which there are no other real points of the curve.




Thus R is a conjugate point. The tangents *at a conjugate point are generally imaginary*, but **sometimes they may be real.**

**SOLVED EXAPLES**

**Example 9.** *Find the nature of the origin on the curve* $a^4y^2 = x^4(x^2 - a^2)$.

**Solution.** Clearly the curve passes through the origin. The equation of the curve can be written as

$$y = \pm \frac{x^2}{a^2}\sqrt{x^2 - a^2}$$

Thus for small values of $x \neq 0$, + ve or – ve, $y$ is imaginary.

∴ In the neighbourhood of (0, 0), no other points of the curve lie and hence origin is a *conjugate point*.

Now
$$\frac{dy}{dx} = \pm \frac{1}{a^2}\left[2x\sqrt{x^2 - a^2} + x^2 \cdot \frac{1}{2} \cdot (x^2 - a^2)^{-1/2} \cdot 2x\right]$$

$$= \pm \frac{1}{a^2}\left[2x\sqrt{x^2 - a^2} + \frac{x^3}{\sqrt{x^2 - a^2}}\right]$$

= 0, at the origin.

∴ Equation of the tangent at the origin (0, 0) is

$$Y - 0 = 0(X - 0) \quad \text{or} \quad Y = 0$$

which is real, **showing that the tangent may be real at a conjugate point.**

### Important Note

The determination of the nature of double points depends basically on the *nature of two branches of the curve passing through it, and not on the tangents to the curve at that point.* Generally, when the tangents at a double point are real, the branches are also real. But there are cases, when the tangents may be real, yet the branches may be imaginary. Thus in such cases our test through the nature of tangents will lead us to wrong results.

## TANGENTS AT THE ORIGIN

**Show that if a curve passing through the origin be given by a rational integral algebric equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the lowest degree terms in the given equation of the curve.**

NOTES

**Proof.** Let the general equation of rational, algebraic curve of $n$th degree, passing through the origin, be

$$(a_1x + b_1y) + (a_2x^2 + b_2xy + c_2y^2) + ...... + (S_{n-1}) + (S_n) = 0 \quad ...(1)$$

where $S_n$ denotes the sum of terms of the $n$th degree in $x$ and $y$ and there is **no constant terms.**

Let P($x$, $y$) be any point on the curve near the origin O.

Slope of chord OP is $\frac{y}{x}$

When P → O, the chord OP → tangent at O

∴ Slope of the tangent at O is $\underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \left(\frac{y}{x}\right) = m$, say.

∴ Equation of the tangent at O is

$$Y = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \left(\frac{y}{x}\right) X = mX \quad ...(2)$$

**Case I. When m is finite i.e., y-axis is not the tangent at the origin.**

(*i*) *Let us suppose* $b_1 \neq 0$.

Dividing both sides of (1) by $x$, we get

$$\left(a_1 + b_1\frac{y}{x}\right) + \left(a_2x + b_2y + c_2y \cdot \frac{y}{2}\right) = ...... = 0$$

When $x \to 0$, $y \to 0$, then Lt $\frac{y}{x} = m$

∴ $a_1 + b_1m = 0$, ∵ all other terms vanish.

or $m = -\frac{a_1}{b_1}$, where $b_1 \neq 0$ ...(3)

∴ (2) becomes $Y = -\frac{a_1}{b_1} X$ or $a_1X + b_1Y = 0$

or taking $x$ and $y$ as current co-ordinates, $a_1x + b_1y = 0$, which could have been obtained by equating to zero the lowest degree terms in (1).

(*ii*) If $b_1 = 0$, then from (3), $a_1 = 0$.

Let us suppose that $b_2 \neq 0$, $c_2 \neq 0$.

Equating (1) becomes

$(a_2x^2 + b_2xy + c_2y^2) + (a_3x^3 + b_3x^2 y + ......) + ...... = 0$

Dividing both sides by $x^2$, we have

$$\left[a_2x + b_2\frac{y}{x} + c_2\left(\frac{y}{x}\right)^2\right] + (a_3x + b_3y + ......) + ...... = 0$$

Let $x \to 0$, $y \to 0$, then Lt $\frac{y}{x} = m$

∴ $a_2 + b_2m + c_2m^2 = 0$ ∵ the other terms vanish. ...(4)

This equation, being a quadratic in $m$, has two values of $m$ and, therefore, there are two tangents at origin.

But $$\frac{Y}{X} = m \qquad ...(5)$$

where $m$ is root of (4).

$\therefore$ Eliminating $m$ between (4) and (5), we have

$$a_2X^2 + b_2XY + c_2Y^2 = 0$$

or taking $x$, $y$ as current co-ordinates,

$$a_2x^2 + b_2xy + c_2y^2 = 0$$

which could have been obtained by equating to zero the lowest degree term of (1).

If $a_1 = b_1 = a_2 = b_2 = c_2 = 0$, it can be similarly shown that the equations of tangents, at the origin, are obtained by equating to zero the terms of lowest degree in the equation of curve; and so on.

**Case II. When Y-axis is tangent at origin.**

$\underset{\substack{x \to 0 \\ y \to -0}}{\text{Lt}} \left(\frac{y}{x}\right)$, being the trignometrical tangent of the inclination of the tangent at the origin to $y$-axis, is zero.

$\therefore$ dividing the equation of curve by $y$ and supposing $a_1 \neq 0$, and making $x$ and $y$ both tend to zero, we find $b_1 = 0$.

Hence equation of curve now being

$a_1x + (a_2x^2 + b_2xy + c_2y^2) + ...... = 0,$

we observe that this theorem is still true in this case also. This proves the proposition.

## WORKING RULE FOR FINDING THE NATURE OF ORIGIN WHICH IS A DOUBLE POINT.

1. *Find the tangents at the origin by equating to zero the lowest degree terms in x and y of the rational, algebraic equation of the curve. If origin is a double point, then we shall get two tangents which may be real or imaginary.*

2. *If two tangents are imaginary, then, origin is a conjugate point.*

3. *If the two tangents are real and distinct, then origin is a node or a conjugate point.*

*To be definite, examine the nature of curve in the neighbourhood of origin. If the curve has real branches through the origin, then it is a node, otherwise a conjugate point.*

4. *If two tangent are real and coincident, then origin is a cusp or a conjugate point. To be definite was test the nature of curve in the neighbourhood of the origin as above in (3).*

### Test for Nature of Curve at Origin

In case the tangents at origin are $y^2 = 0$, solvethe equation of curve for $y$, neglecting all terms of $y$ containing powers above second. If the values of $y$, for small values of $x$, are found to be real, the branches of curve through the origin are real, otherwise imaginary.

If the tangents at origin are $x^2 = 0$, solve the equation for $x$ and proceed as above.

The students must not that while neglecting higher powers of $x$ and $y$, the reduced equation may not coincide with that of the tangents or the two branches of curve may not coincide.



If the tangents at origin are $x^2 = 0$, solve the equation for $x$ and proceed as above.

The students must note that while neglecting higher powers of $x$ and $y$, the redued equation may not coincide with that of the tangents or the two branches of curve may not coincide.

If we are to study the nature of double point, which is not origin, we transfer the origin to the double point, say, $(h, k)$ and proceed as above.

## SOLVED EXAMPLES

**Example 10.** *Find the nature of origin on the following curves:*

(*i*) $x^4 - ax^2y + axy^2 + a^2y^2 = 0$ (*ii*) $y^2 = 2x^2y + x^4y - 2x^4$.

**Solution.** (*i*) The given equation of the curve is

$$x^4 - ax^2y + axy^2 + a^2y^2 = 0$$

Equating to zero, the lowest degree terms, the tangents at the origin are given by $a^2y^2 = 0$ or $y^2 = 0$ *i.e.*, $y = 0, y = 0$.

∴ There are two real and coincident tangents at the origin.

∴ Origin is either a **cusp or a conjugate point.**

From (1), $ay^2(x + a) - ax^2y + x^4 = 0$

or
$$y = \frac{ax^2 \pm \sqrt{a^2x^4 - 4ax^4(x+a)}}{2a(x+a)}$$

$$= \frac{ax^2 \pm x^2\sqrt{-4ax - 3a^2}}{2a(x+a)}$$

For small values of $x \neq 0$, $(-4ax - 3a^2)$ is – ve ∴ $y$ is imaginary in the neighbourhood of origin.

Hence origin is a **conjugate point.**

(*ii*) The equation of the curve is $y^2 = 2x^2y + x^4y - 2x^4$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by $y^2 = 0$ or $y = 0, y = 0$.

There are two real and coincident tangent at the origin.

∴ Origin is either a *cusp or a conjugate point.*

From (1), $y^2 - x^2y(2 + x^2) + 2x^4 = 0$.

Solving for $y$,
$$y = \frac{x^2(2+x^2) \pm \sqrt{x^4(2+x^2)^2 - 8x^4}}{2}$$

$$= \frac{x^2(x^2+2) \pm x^2\sqrt{x^4 + 4x^2 - 4}}{2}$$

When $x$ is small $\neq 0$, $x^4 + 4x^2 - 4$ is –ve, so that $y$ is imaginary in the neighbourhood of origin.

$\therefore$ Origin is a conjugate point.

**Note.** These are examples of the case when the double point is a conjugate point, even though the tangents are real.

**Example 11.** *Find the nature of the origin for the following curves:*

(*i*) $y^3 = x^3 + ax^2$ (*ii*) $x^2(x - y) + y^2 = 0$.

**Solution.** (*i*) The equation of the curve is

$$y^3 = x^3 + ax^2 \quad \ldots(i)$$

Equating to zero, the lowest degree terms in (*i*), the tangent at the origin are given by $x^2 = 0$ or $x = 0, x = 0$.

Since the two tangents are real and coincident, the origin is a *crusp or a conjugate point.*

Neglecting $x^3$ in (*i*), we have $ax^2 = y^3$ or $x = \pm\sqrt{\frac{y}{a}}$

Supposing $a > 0$, $x$ is real for small + ve values of $y$.

Hence the two branches of the curve near the origin are real and so the origin is a cusp.

(*ii*) The equation of given curve is $x^3 - x^2y + y^2 = 0$ ...(1)

Equating the zero, the lowest degree terms, the tangents at the origin are $y^2 = 0$ or $y = 0, y = 0$.

$\because$ the two tangents are real and coincident, $\therefore$ origin is a cusp or conjugate point.

Solving (1) for $y$, we have

$$y = \frac{x^2 \pm \sqrt{x^4 - 4x^3}}{2} = \frac{x^2 \pm \sqrt{x^3(x - 4)}}{2}$$

When $x$ is small and – ve, $x^3(x - 4)$ is + ve and so $y$ is real in the neighbourhood or origin. [**Note** that $y$ is imaginary for $x > 0$ near the origin].

Thus there are two real branches of the curve near the origin (for < 0),

$\therefore$ Origin is a cusp.

**Example 12.** *Show that (3, 1) is a cusp on the curve*

$$(y - 1)^2 = (x - 3)^3.$$

**Solution.** The equation of the curve is

$$(y - 1)^2 = (x - 3)^3 \quad \ldots(1)$$

Shifting the origin to (3, 1) [by putting $x = X + 3$, $y = Y + 1$], (1) reduces to

$$(Y + 1 - 1)^2 = (X + 3 - 3)^3 \quad \text{or} \quad Y^2 = X^3 \quad \ldots(2)$$

Equating to zero the lowest degree terms, the tangents at the new origin are given by

$$Y^2 = 0, \quad \text{or} \quad Y = 0, Y = 0$$

Since the two tangents are real and coincident, $\therefore$ new origin is a cusp or conjugate point.

From (2), $Y = \pm X\sqrt{X}$, which gives real values of Y for small + ve value of X. Hence, real branches of the curve exist in the neighbourhood of the new origin.

$\therefore$ The new origin *i.e.*, the point (3, 1) is a cusp.

### Show that the Necessary and Sufficient Conditions for any Point (x, y) on f(x, y) = 0 to be a Multiple Point are that $f_x(x, y) = 0$, $f_y(x, y) = 0$

The equation of the curve is $f(x, y) = 0$

Differential (1) w.r.t. $x$, treating $y$ as a function of $x$, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad ...(2)$$

where $\frac{dy}{dx}$ gives the slope of the tangent at the point P($x$, $y$).

If P($x$, $y$) is a multiple point, there must be at least two tangents which may be real, coincident or imaginary.

Thus $\frac{dy}{dx}$ must have at least two values at ($x$, $y$). But eqaution (2) is a first degree equation in $\frac{dy}{dx}$ and is satisfied by at least two values of $\frac{dy}{dx}$, which it possible only if it becomes an identity.

Thus $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0.$

Also ($x$, $y$) lies on the given curve $f(x, y) = 0$.

∴ *The necessary and sufficient conditions for any point (x, y) on the curve f(x, y) = 0 to be a multiple point is*

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0 \qquad ...(3)$$

*i.e.* **$f_x(x, y) = 0$ and $f_y(x, y) = 0$.**

Hence the result.

## CLASSIFICATION OF DOUBLE POINTS

*The simultaneous solutions of equations (3) which* **also** *satisfy the equation (1) give the positions of double points or multiple points on the curve.*

Differentiating (2) w.r.t. $x$, we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dx} + \left( \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dx} \right) \cdot \frac{dy}{dx} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dx^2} = 0$$

But $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ and also at the double point

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

∴ The above equation reduces to

$$\frac{\partial^2 f}{\partial y^2} \left( \frac{dy}{dx} \right)^2 + 2 \frac{\partial^2 f}{\partial y \partial x} \cdot \frac{dy}{dx} + \frac{\partial^2 f}{\partial x^2} = 0 \qquad ...(4)$$

which is quadratic in $\frac{dy}{dx}$ and gives the two slopes of the tangents at the double point ($x$, $y$).

NOTES

The tangents at $(x, y)$ will be real and distinct, real and coincident or imaginary according as the roots of (4) are real and distinct, equal or imaginary for which

$$4\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - 4\,\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > 0, = 0 \quad \text{or} \quad < 0$$

or
$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > 0, = 0 \quad \text{or} \quad < 0$$

Hence **in general,** *a double point will be a node, cusp or a conjugate point, according as*

$$\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x} \partial \mathbf{y}}\right)^2 >, \quad \text{or} \quad < \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \cdot \frac{\partial^2 \mathbf{f}}{\partial \mathbf{y}^2} \text{ at that point.} \qquad \text{...(5)}$$

**Note. 1.** If $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y \partial x} = 0$ at the point $(x, y)$, then the point $(x, y)$ will be a multiple point of order higher than two.

**Note 2.** The condition (5) is not a sure test for the node, cusp or a conjugate point. This in fact is the condition for the two tangents at the double point to be real and distinct, coincident or imaginary. The result may sometimes lead to the wrong conclusion. [*See Example 1 below*]

## Working Rule for Finding the Position and Nature of Double Points of the Curve f(x, y) = 0.

1. *Find the position of double points from the equations*

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \quad \text{and} \quad f(x, y) = 0.$$

2. *Find the nature by shifing the origin to the double points and then testing the nature of tangents and the existence of the curve in the neighbourhood of new origin.*

## SOLVED EXAMPLE

**Example 13.** *Find the position and nature of the double points of the following curves:*

(*a*) $x^3 + x^2 + y^2 - x - 4y + 3 = 0$

(*b*) $2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0$

(*c*) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$

**Solution.** (*a*) The equation of the curve is

$$f(x, y) \equiv x^3 + x^2 + y^2 - x - 4y + 3 = 0 \qquad \text{...(1)}$$

$$\therefore \quad \left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 + 2x - 1 = (3x - 1)(x + 1) \\ \frac{\partial f}{\partial y} &= 2y - 4 = 2(y - 2) \end{aligned}\right\} \qquad \text{...(2)}$$

Now for the double points $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

NOTES

$$\frac{\partial f}{\partial x} = 0 \text{ gives } (3x - 1)(x + 1) = 0 \quad \therefore \quad x = \frac{1}{3} \text{ or } -1$$

and $$\frac{\partial f}{\partial y} = 0 \text{ gives } 2(y - 2) = 0 \quad \therefore \quad y = 2$$

$\therefore$ The possible double points are $\left(\frac{1}{3}, 2\right)$ and $(-1, 2)$.

Of these, only $(-1, 2)$ satisfies the given equation (1) of the curve.

Hence $(-1, 2)$ is the only double point of the curve.

Shifting the origin to $(-1, 2)$ [by putting $x = X - 1$ and $y = Y + 2$ in (1)], the equation (1) reduces to

$$(X - 1)^3 + (X - 1)^2 + (Y + 2)^2 - (X - 1) - 4(Y + 2) + 3 = 0$$

or $$X^3 - 3X^2 + 3X - 1 + X^2 - 2X + 1 + Y^2 + 4Y + 4 - X + 1 - 4Y - 8 + 3 = 0$$

or $$X^3 - 2X^2 + Y^2 = 0 \qquad ...(3)$$

The tangents at the new origin are given by $-2X^2 + Y^2 = 0$ or $Y = \pm\sqrt{2}X$. Since the two tangents are real and distinct, new origin is a node or a conjugate point.

Solving (3) for Y, we get $Y = \pm X\sqrt{2 - X}$

Now for small values of X, + ve or – ve, Y is real. $\therefore$ Two real branches of the curve pass through $(-1, 2)$. Hence $(-1, 2)$ is a node.

(*b*) The equation of the curve is

$$f(x, y) \equiv 2(x^3 + y^3) - 3(3x^2 + y^2) + 12x - 4 = 0 \qquad ...(1)$$

$\therefore$
$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2) \\ \frac{\partial f}{\partial y} &= 6y^2 - 6y = 6y(y - 1) \end{aligned}\right] \qquad ...(2)$$

Now for the double points, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \text{ gives } 6(x - 1)(x - 2) = 0 \quad \therefore \quad x = 1 \text{ or } 2.$$

$$\frac{\partial f}{\partial y} = 0 \text{ gives } 6y(y - 1) = 0 \quad \therefore \quad y = 0 \text{ or } 1.$$

Thus the possible double points are (1, 0), (2, 0), (1, 1), (2, 1)

Of these only (1, 1) and (2, 0) satisfy the given equation (1) of the curve and hence these are the only double points.

*Nature of the point* (1, 1).

Shifting the origin to the point (1, 1) [by putting $x = X + 1$, $y = Y + 1$], the equation (1) transforms to

$$2(X + 1)^3 + 2(Y + 1)^3 - 9(X + 1)^2 - 3(Y + 1)^2 + 12(X + 1) - 4 = 0$$

or $$2(X^3 + 3X^2 + 3X + 1) + 2(Y^3 + 3Y^2 + 3Y + 1) - 9(X^2 + 2X + 1)$$
$$- 3(Y^2 + 2Y + 1) + 12X + 12 - 4 = 0$$

or $$2X^3 + 2Y^3 - 3X^2 + 3Y^2 = 0 \qquad ...(3)$$

Equating to zero the lowest degree terms in (3), the tangents at the new origin are $3Y^2 - 3X^2 = 0$ or $Y = \pm X$.

Since these tangents are real and distinct, the new origin is a node or a conjugate point.

Solving (3) for Y(and neglecting $Y^3$ and higher powers of Y), we get

$$3Y^2 = 3X^2 - 2X^3 \quad \text{or} \quad Y = \pm X\sqrt{1 - \frac{2}{3}X}.$$

For small values of X, + ve or – ve, Y is real. Hence two real branches of the curve pass through the new origin $\therefore$ (1, 1) is a node.

*Nature of the point* (2, 0).

Shifting the origin to the point (2, 0) [by putting $x = X + 2$, $y = Y + 0$], the equation (1) transform to

$$2(X + 2)^3 + 2Y^3 - 9(X + 2)^2 - 3Y^2 + 12(X + 2) - 4 = 0$$

or $$2(X^3 + 6X^2 + 12X + 8) + 2Y^3 - 9(X^2 + 4X + 4) - 3Y^2 + 12X + 24 - 4 = 0$$

or $$2X^3 + 2Y^3 + 3X^2 - 3Y^2 = 0 \quad \text{...(4)}$$

Equating to zero the lowest degree terms, the tangents at the new origin are given by

$$3X^2 - 3Y^2 = 0 \quad \text{or} \quad Y^2 = X^2 \quad \therefore \quad Y = \pm X.$$

Since the tangents are real and distinct, the new origin is a node or a conjugate point.

Solving (4) for Y(and neglecting $Y^3$ and higher powers of Y), we get

$$3Y^2 = 3X^2 + 2X^3 \quad \text{or} \quad Y = \pm X\sqrt{1 - \frac{2}{3}X}$$

Now for small values of X, + ve or – ve, Y is real.

$\therefore$ Two real branches of the curve pass through the new origin, *i.e.*, the point (2, 0). Hence the point (2, 0) is a node.

Thus the curve has two nodes at (1, 1) and (2, 0).

(*c*) The equation of the curve is

$$f(x, y) \equiv x^3 + 3x^2y - 4y^3 - x + y + 3 = 0 \quad \text{...(1)}$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 + 6xy - 1; \quad \frac{\partial f}{\partial y} = 3x^2 - 12y^2 + 1 \quad \text{...(2)}$$

Now for the double points, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \text{ gives } \quad 3x^2 + 6xy - 1 = 0 \quad \text{...(3)}$$

$$\frac{\partial f}{\partial y} = 0 \text{ gives } \quad 3x^2 - 12y^2 + 1 = 0 \quad \text{...(4)}$$

From (3), $y = \frac{1 - 3x^2}{6x}$ putting in (4), we get

$$3x^2 - 12 . \left(\frac{1 - 3x^2}{6x}\right)^2 + 1 = 0 \quad \text{or} \quad 3x^2 - \frac{(1 - 3x^2)^2}{3x^2} + 1 = 0$$

or $$9x^4 - (1 + 9x^4 - 6x^2) + 3x^2 = 0$$

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or $$9x^2 - 1 = 0 \quad \text{or} \quad x^2 = \frac{1}{9} \quad \text{or} \quad x = \pm\frac{1}{3}$$

Putting this value of $x$ in (4), $3\left(\frac{1}{9}\right) - 12y^2 + 1 = 0$

or $$12y^2 = \frac{1}{3} + 1 = \frac{4}{3} \quad \therefore \quad y^2 = \frac{1}{9} \text{ or } y = \pm\frac{1}{3}.$$

Hence the possible double points are

$$\left(\frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, -\frac{1}{3}\right)\left(-\frac{1}{3}, \frac{1}{3}\right) \text{ and } \left(-\frac{1}{3}, -\frac{1}{3}\right).$$

Out of these, none satisfies the equation (1). Hence there is no double point.

**Example 14.** *Find the position and nature of the double points of the following curves:*

(*a*) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$

(*b*) $x^4 - 4y^3 - 12y^2 - 8x^2 + 16 = 0$.

**Solution.** (*a*) The equation of the given curve is

$$f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0 \quad \ldots(1)$$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 - 14x + 15 \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y + 4$$

For the double points $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \text{ gives } 3x^2 - 14x + 15 = 0 \quad \text{or} \quad (x - 3)(3x - 5) = 0$$

$$\therefore \quad x = 3, \frac{5}{3}.$$

$$\frac{\partial f}{\partial y} = 0 \text{ gives } -2y + 4 = 0 \quad \therefore \quad y = 2.$$

$\therefore$ The possible double points are (3, 2) and $\left(\frac{5}{3}, 2\right)$.

But out of these only (3, 2) satisfies the given equation (1).

$\therefore$ (3, 2) is the only double point.

*Nature of the point* (*3, 2*).

Shifting the origin to the point (3, 2), [by putting $x = X + 3$, $y = Y + 2$].

(1) transforms to

$$(X + 3)^3 - (Y + 2)^2 - 7(X + 3)^2 + 4(Y + 2) + 15(X + 3) - 13 = 0$$

or $$(X^3 + 9X^2 + 27X + 27) - (Y^2 + 4Y + 4) - 7(X^2 + 6X + 9) + 4Y + 8 + 15X + 45 - 13 = 0$$

or $$X^3 + 2X^2 - Y^2 = 0 \quad \ldots(2)$$

Equating to zero the lowest degree terms, the tangents at the new origin are given by $2X^2 - Y^2 = 0$ or $Y = \pm\sqrt{2}X$.

Since the two tangents are real and distinct, $\therefore$ new origin is a node or a conjuate point.

Solving (2) for Y, we get $Y = \pm X\sqrt{2 + X}$

which gives real values of Y for small enough values of X, + ve, or – ve.

$\therefore$ Real branches of the curve exist in the neighbourhood of the new origin.

$\therefore$ new origin *i.e.* the point (3, 2) is a node.

Hence the given curve has a node at the point (3, 2).

(*b*) The equation of the curve is

$$f(x, y) = x^4 - 4y^3 - 12y^2 - 8x^2 + 16 = 0 \qquad ...(1)$$

$$\therefore \qquad \frac{\partial f}{\partial x} = 4x^3 - 16x = 4x(x^2 - 4)$$

and
$$\frac{\partial f}{\partial y} = -12y^2 - 24y = -12y(y + 2)$$

Now for the double points, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 0$ gives $x = 0, \pm 2$ and $\frac{\partial f}{\partial y} = 0$ gives $y = 0, -2$.

$\therefore$ The possible doubles points are (0, 0), (0, – 2), (2, 0), (2, – 2), (– 2, 0) and (– 2, – 2).

Out of these, only (0, – 2), (2, 0) and (– 2, 0) satisfy the given equation (1).

*Nature of the point (0, – 2)*

Shifting the origin to the point (0, – 2),

[by putting $x = X + 0 = X$, $y = Y - 2$], (1) transforms to

$$X^4 - 4(Y - 2)^3 - 12(Y - 2)^2 - 8X^2 + 16 = 0$$

or
$$X^4 - 4(Y^3 - 6Y^2 + 12Y - 8) - 12(Y^2 - 4Y + 4) - 8X^2 + 16 = 0$$

or
$$X^4 - 4Y^3 + 12Y^2 - 8X^2 = 0 \qquad ...(2)$$

Equating to zero the lowest degree terms, the tangents at the new origin are

given by
$$12Y^2 - 8X^2 = 0 \quad \text{or} \quad Y = \pm \sqrt{\frac{2}{3}}X$$

which are real and distinct. $\therefore$ The new origin is a node or a conjugate point.

Solving (2) for Y (neglecting $Y^3$), we get $12Y^2 = X^2(8 - X^2)$

or
$$2\sqrt{3}\,Y = \pm X\sqrt{8 - X^2}$$

For small value of X, + ve or – ve, Y is real. $\therefore$ Real branches of the curve exist in the neighbourhood of the new origin.

Hence the new origin *i.e.* the point (0, – 2) is a node.

*Nature of point (0, – 2)*

Shifting the origin to the point (2, 0),

[by putting $x = X + 2$, $y = Y + 0 = Y$], (1) transforms to

$$(X + 2)^4 - 4Y^3 - 12Y^2 - 8(X + 2)^2 + 16 = 0$$

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or $(X^4 + 8X^3 + 24X^2 + 32X + 16) - 4Y^3 - 12Y^2 - 8(X^2 + 4X + 4) + 16 = 0$

or $X^4 + 8X^3 - 4Y^3 + 16X^2 - 12Y^2 = 0$ ...(2)

Equating to zero the lowest degree terms, the tangents at the new origin are given by

$$12Y^2 - 8X^2 = 0 \quad \text{or} \quad Y = \pm \sqrt{\frac{2}{3}}X$$

which are real and distinct. $\therefore$ The new origin is a node or a conjugate point.

Solving (2) for Y(neglecting $Y^3$), we get $12Y^2 = X^2(8 - X^2)$

or $$2\sqrt{3}\ Y = \pm \sqrt{8 - X^2}$$

For small values of X, +ve or – ve, Y is real. $\therefore$ Real branches of the curve exist in the neighbourhood of the new origin.

Hence the new origin *i.e.* the point (0, – 2) is a node.

*Nature of the point (2, 0)*

Shifting the origin to the point (2, 0), [by putting $x = X + 2$, $y = Y + 0 = Y$], (1) transforms to

$$(X + 2)^4 - 4Y^3 - 12Y^2 - 8(X + 2)^2 + 16 = 0$$

or $(X^4 + 8X^3 + 24X^2 + 32X + 16) - 4Y^3 - 12Y^2 - 8(X^2 + 4X + 4) + 16 = 0$

or $X^4 + 8X^3 - 4Y^3 + 16X^2 - 12Y^2 = 0$ ...(3)

Equating to zero the lowest degree terms, the tangents at the new origin are given by

$$16X^2 - 12Y^2 = 0 \quad \text{or} \quad Y = \pm \sqrt{\frac{4}{3}}\ X$$

which are real and distinct. $\therefore$ The new origin is a node or a conjugate point.

Solving (3) for Y(neglecting $Y^3$), we get

$$12Y^2 = X^4 + 8X^3 + 16X^2 = X^2(X + 4)^2$$

or $Y = \pm \dfrac{1}{2\sqrt{3}} X(X + 4)$. Thus Y is real for all values of X.

$\therefore$ Real branches of the curve exist in the neighbourhood of the new origin.

Hence the new origin *i.e.*, the point (2, 0) is a node.

Similarly (– 2, 0) is a node. [Left as an exercise for the student.]

## EXERCISE – 2

1. Show that the curve $r = a \cdot \dfrac{\theta^2}{\theta^2 - 1}$ has a point of inflexion at $r = \dfrac{3a}{2}$.

2. Find the nature of the origin for the following curves:

   (*i*) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ (*ii*) $x^4 + y^4 - 4axy = 0$

   (*iii*) $\dfrac{a^2}{x^2} - \dfrac{b^2}{y^2} = 1.$

NOTES

3. (*i*) Show that the curve $y^2 = bx \sin \frac{x}{a}$ has a node or a conjugate point at the origin, according as $a$ and $b$ have like or unlike signs.

(*ii*) Show that the curve $y^2 = 2x \sin x$ has a node at the origin.

(*iii*) Show that the curve $y^2 = bx \tan \frac{x}{a}$ has a node or a conjugate point at the origin, according as $a$ and $b$ have like or unlike signs.

(*iv*) Show that the origin is a node, a cusp or a conjugate point on the curve $y^2 = ax^2 + bx^3$, according as a is positive, zero or negative.

4. Find the position and nature of the double points on the curve $a^4y^2 = x^4(2x^2 - 3a^2)$.

5. Find the position and nature of the double points on the following curves:

(*a*) $y^2 = 2x^2y + x^3y + x^3$ (*b*) $x^3 + y^3 - 3axy = 0$

(*c*) $x(x^2 + y^2) - ay^2 = 0$.

6. Find the position and nature of the double points on the following curves:

(*i*) $(y - x)^2 + x^6 = 0$ (*K.U. 1986 ; M.D.U. 1982, 89 S*)

(*ii*) $(x - y)^2 + x^4 = 0$. (*K.U. 1981 S*)

7. (*i*) Prove that the curve $y^2 = (x - a)^2 (x - b)$ has at $x = a$, $a$ conjugate point if $a < b$, $a$ node if $a > b$ and $a$ cusp if $a = b$. (*D.U. 1987*)

8. Find the position and nature of the double points on the following curves:

(*a*) $(x - 2)^2 = y(y - 1)^2$. (*K.U. 1984, 89*)

(*b*) $y^2 = (x - 1)(x - 2)^2$ (*D.U. 1990; M.D.U. 1985*)

(*c*) $(y - 2)^2 = x(x - 1)^2$.

9. Show that each of the curves

$$(x \cos \alpha - y \sin \alpha - b)^3 = c(x \sin \alpha + y \cos \alpha)^2$$

for all different values of $\alpha$, has $a$ cusp; show also that all the cusps lie on a circle.

**Answers**

2. (*i*) node. (*ii*) Origin is a node.

(*iii*) Node

4. Origin is a *conjugate point and not a cusp.*

5. (*a*) The given curve has a cusp at the origin.

(*b*) The curve has a node at the origin.

(*c*) The given curve has a cusp at the origin.

6. (*i*) The origin is a conjugate point.

(*ii*) Conjugate point

8. (*a*) node at the point (2, 1).

(*b*) node at (2, 0).

(*c*) node at (1, 2)

## ANOTHER METHOD OF FINDING THE POSITION OF DOUBLE POINTS

Let $(h, k)$ be the double point on the curve $f(x, y) = 0$. Transfer the origin to $(h, k)$ by the substitution $x = X + h$, $y = Y + k$, and let the transformed equation be $F(X, Y) = 0$. Since the new origin is a double point, the constant term and the terms of the first degree in $F(X, Y) = 0$ must be absent.

$\therefore$ Equating to zero the constant term, co-efficients of X and Y separately to zero in F(X, Y) = 0, we get three equations in ($h$, $k$). Solving any two of these equations for $h$ and $k$ and if these values of $h$, $k$ also satisfy the third equation, then ($h$, $k$) is a double point.

NOTES

## SOLVED EXAMPLES

**Example 15.** *Determine the position and character of the double points on the following curves:*

($a$) $y(y - 6) = x^2 (x - 2)^3 - 9$ ($b$) $(2y + x + 1)^2 - 4(1 - x)^5 = 0$

($c$) $(x + y)^3 - \sqrt{2}\,(y - x + 2)^2 = 0.$

**Solution.** ($a$) The given equation of the curve is

$$f(x, y) \equiv x^2(x - 2)^3 - y(y - 6) - 9 = 0 \qquad ...(1)$$

$$\therefore \quad \frac{\partial f}{\partial x} = 2x(x - 2)^3 + 3(x - 2)^2 \,.\, x^2 = x(x - 2)^2\,[2(x - 2) + 3x]$$

$$= x(x - 2)^2\,(5x - 4)$$

$$\frac{\partial f}{\partial y} = -\,2y + 6$$

For the double points, $\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0.$

$$\frac{\partial f}{\partial x} = 0 \text{ gives } x(x - 2)^2\,(5x - 4) = 0 \quad \text{or} \quad x = 0, 2, \frac{4}{5}$$

$$\frac{\partial f}{\partial y} = 0 \text{ gives } -\,2y + 6 = 0 \qquad \text{or} \quad y = 3.$$

$\therefore$ The possible double points are (0, 3), (2, 3) and $\left(\frac{4}{5}, 3\right)$.

Out of these, only (0, 3) and (2, 3) satisfy (1).

$\therefore$ (0, 3) and (2, 3) are the only double points.

*Character of (0, 3).*

Shifting the origin to (0, 3) by putting $x = X + 0$, $y = Y + 3$, the given equation (1) transforms to

$$(Y + 3)\,(Y + 3 - 6) = X^2(X - 2)^3 - 9$$

or $$Y^2 - 9 = X^2(X - 2)^3 - 9 \quad \text{or} \quad Y^2 = X^2(X - 2)^3 \qquad ...(2)$$

Equating to zero, the lowest degree terms, the tangents at the new origin are $Y^2 + 8X^2 = 0$, which are imaginary. $\therefore$ new origin *i.e.* (0, 3) is a conjugate point.

*Character of (2, 3).*

Shifting the origin to (2, 3) by putting $x = X + 2$, $y = Y + 3$, the equation (1) transforms to

$$(Y + 3)\,(Y + 3 - 6) = (X + 2)^2\, X^3 - 9$$

or $$Y^2 = X^3(X + 2)^2 \qquad ...(3)$$

Equating to zero the lowest degree terms, the tangents at the new origin are $Y^2 = 0$ or $Y = 0$, $Y = 0$, which being real and coincident, new origin is a cusp or a conjugate point.

But from (3), $Y = \pm X(X + 2)\sqrt{X}$ which determines real values of Y for all + ve values of X. $\therefore$ Real branches of the curve exist in the neighbourhood of the new origin.

Hence the new origin *i.e.*, (2, 3) is a cusp.

Thus the given curve has a cusp at (2, 3) and a conjugate point at (0, 3).

(*b*) The equation of the given curve is

$$f(x, y) \equiv (2y + x + 1)^2 - 4(1 - x)^5 = 0 \qquad ...(1)$$

$$\therefore \qquad \frac{\partial f}{\partial x} = 2(2y + x + 1) + 20(1 - x)^4$$

and $$\frac{\partial f}{\partial y} = 4(2y + x + 1).$$

For double points, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$$\therefore \qquad \frac{\partial f}{\partial x} = 0 \text{ gives } 2(2y + x + 1) + 20(1 - x)^4 = 0 \qquad ...(2)$$

and $$\frac{\partial f}{\partial y} = 0 \text{ gives } 4(2y + x + 1) = 0 \quad \text{or} \quad 2y + x + 1 = 0 \qquad ...(3)$$

Using (3), (2) becomes, $20(1 - x)^4 = 0$ or $1 - x = 0$ $\therefore$ $x = 1$.

Putting $x = 1$, in (3), we get $2y + 1 + 1 = 0$ or $y = -1$

Thus we get the point (1, – 1) satisfies the given equation (1) of the curve. $\therefore$ (1, – 1) is a double point.

*Character of (1, – 1).*

Shifting the origin to (1, – 1) by putting $x = X + 1$, $y = Y - 1$, the equation (1) transforms to

$$[2(Y - 1) + (X + 1) + 1]^2 - 4[1 - (X + 1)]^5 = 0$$

or $$(2Y + X)^2 - 4X^5 = 0 \qquad ...(4)$$

Equating to zero, the lowest degree terms, the tangents at the new origin are $(2Y + X)^2 = 0$, which are real and coincident. Hence the new origin is a cusp or a conjugate point.

Now from (4), $2Y + X = \pm\, 2X^2\sqrt{-X}$, which gives real values of Y for – ve values of X. Hence real branches of the curve pass through the new origin

$\therefore$ the new origin *i.e.*, (1, – 1) is a cusp.

Hence the curve has a cusp at the point (1, – 1).

(*c*) The equation of the given curve is

$$f(x, y) \equiv (x + y)^3 - \sqrt{2}\,(y - x + 2)^2 = 0 \qquad ...(1)$$

For the double points $\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 3(x + y)^2 + 2\sqrt{2}\,(y - x + 2) = 0 \qquad ...(2)$$

$$\frac{\partial f}{\partial y} = 3(x + y)^2 - 2\sqrt{2}\,(y - x + 2) = 0 \qquad ...(3)$$

Adding and subtracting (2) and (3), we get

$$6(x + y)^2 = 0 \quad \text{or} \quad x + y = 0 \qquad ...(4)$$

and $$4\sqrt{2}\,(y - x + 2) = 0 \quad \text{or} \quad y - x + 2 = 0 \qquad ...(5)$$



We get the point (1, – 1), Also (1, – 1) satisfies (1).

∴ (1, – 1) is a double point.

*Character of (1, – 1).*

Shifting the origin to (1, – 1), by putting $x = X + 1$, $y = Y - 1$, the equation (1) transforms to

$$[(X + 1) + (Y - 1)]^3 - \sqrt{2}\,[(Y - 1) - (X + 1) + 2]^2 = 0$$

or $$(X + Y)^3 - \sqrt{2}\,(Y - X)^2 = 0 \qquad ...(6)$$

Equation to zero the lowest degree terms, the tangents at the new origin are $(Y - X)^2 = 0$, which are real and coincident new origin is a cusp or a conjugate point.

From (6) neglecting $Y^3$ and higher power of Y, we get

$$X^3 + 3X^2Y + 3XY^2 - \sqrt{2}\,(Y^2 - 2XY + X^2) = 0$$

or $$Y^2(3X - \sqrt{2}) - XY(2\sqrt{2} - 3X) - X^2(\sqrt{2} - X) = 0$$

$$\therefore \quad Y = \frac{X(2\sqrt{2} - 3X) \pm \sqrt{X^2(2\sqrt{2} - 3X)^2 + 4X^2(\sqrt{2} - X)(3X - \sqrt{2})}}{2(3X - \sqrt{2})}$$

$$= \frac{X(2\sqrt{2} - 3X) \pm X\sqrt{(8 + 9X^2 - 12\sqrt{2}X) + 4(-2 - 3X^2 + 4\sqrt{2}X)}}{2(3X - \sqrt{2})}$$

$$= \frac{X(2\sqrt{2} - 3X) \pm X\sqrt{4\sqrt{2}X - 3X^2}}{2(3X - \sqrt{2})}$$

When X is small and + ve, $4\sqrt{2}X - 3X^2$ is + ve.

∴ Y is real. Hence real branches of the curve through the new origin. Thus the new origin *i.e.*, (1, – 1) is cusp.

Hence, the curve has a cusp of the point (1, – 1).

**Example 16.** (*a*) *Examine the curve $a^2y^2 = a^2x^2 - 4x^3$ for singular points.*

(*b*) *Show that for the curve $y^2 = (x - 2)^2(x - 5)$, the straight line joinning the points of inflexion subtends a right angle at the double point.*

**Solution.** (*a*) The eqaution of the curve is $a^2y^2 = x^2(a^2 - 4x)$ ...(1)

[**Note that,** singular points mean the points of inflexion and the multiple points. Thus, we have to examine the curve (1) for the points of inflexion and the double points.]

*For the point of inflextion:*

From (1), $$ay = \pm\, x\sqrt{a - 4x}$$

$$\therefore \quad a\frac{dy}{dx} = \pm\left[\sqrt{a - 4x} + x \cdot \frac{1(-4)}{2\sqrt{a^2 - 4x}}\right]$$

$$= \pm\,\frac{a^2 - 4x - 2x}{\sqrt{a^2 - 4x}} = \pm\,(a^2 - 6x)(a^2 - 4x)^{-1/2}$$

and $$a\frac{d^2y}{dx^2} = \pm\,[(a^2 - 6x)\left(-\tfrac{1}{2}\right)(a^2 - 4x)^{-3/2}\,.\,(-4) - 6(a^2 - 4x)^{-1/2}]$$

$$= \pm\, 2(a^2 - 4x)^{-3/2}[(a^2 - 6x) - 3(a^2 - 4x)]$$

$$= \pm\, 2(a^2 - 4x)^{-3/2}\,[6x - 2a^2]$$

$$\therefore \quad a\frac{d^3y}{dx^3} = \pm\, 2[6(a^2 - 4x)^{-3/2} + (6x - 2a^2)\left(-\tfrac{3}{2}\right)(a^2 - 4x)^{-5/2}\,.\,(-4)]$$

$$= \pm\, 12(a^2 - 4x)^{-5/2}\,[(a^2 - 4x) + 6x - 2a^2]$$

$$= \pm\,\frac{12(2x - a^2)}{(a^2 - 4x)^{5/2}}.$$

For a point of inflexion, $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

$\therefore \quad \frac{d^2y}{dx^2} = 0$ gives $6x - 2a^2 = 0$ or $x = \frac{a^2}{3}$

Also $\quad x = \frac{a^2}{3}$ gives $\frac{d^3y}{dx^3} \neq 0$

But when $x = \frac{a^2}{3}$, from (1),

$$a^2y^2 = \frac{a^4}{9}\left(a^2 - \frac{4a^2}{3}\right) = -\text{ve} \quad \therefore \quad y \text{ is imaginary.}$$

Hence the curve has **no point of inflexion.**

*For the multiplied points:*

The equation (1) can be written as

$$f(x, y) = 4x^3 - a^2x^2 + a^2y^2 = 0 \qquad \ldots(2)$$

$$\frac{\partial f}{\partial x} = 12x^2 - 2a^2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2a^2y.$$

For multiple points, $\frac{\partial f}{\partial x} = 0,\ \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \text{ gives } 2x(6x - a^2) = 0 \quad \text{or} \quad x = 0,\ \frac{a^2}{6}$$

$$\frac{\partial f}{\partial y} = 0 \text{ gives } y = 0$$

$\therefore$ The possible multiple points are (0, 0), $\left(\frac{a^2}{6}, 0\right)$

But only (0, 0) satisfies the given curve (1).

$\therefore$ (0, 0) is the only multiple point on the curve.

*Nature of origin.* Equating to zero, the lowest degree terms in (2), the tangents at the origin are given by

$$-a^2x^2 + a^2y^2 = 0 \quad \text{or} \quad y^2 = x^2 \quad \therefore \quad y = \pm\, x$$

Since the two tangent are real distinct, $\therefore$ origin is a node or a conjugate point.

NOTES

From (2), $ay = \pm x\sqrt{a^2 - 4x}$, which gives real values of $y$ for small values of $x$, + ve or – ve.

$\therefore$ Real branches of the curve pass through the origin.

Hence the origin is a node.

Thus the curve has **a node at the origin.**

(*b*) The equation of the curve is $y^2 = (x - 2)^2 (x - 5)$ ...(1)

**Points of inflexion:**

From (1), we have $y = \pm (x - 2)\sqrt{x - 5}$

$$\therefore \quad \frac{dy}{dx} = \pm \left[1 . \sqrt{x-5} + (x-2) . \frac{1}{2\sqrt{x-5}}\right]$$

$$= \pm \left[\frac{2(x-5) + x - 2}{2\sqrt{x-5}}\right] = \pm \frac{1}{2}(3x - 12)(x-5)^{-1/2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \pm \frac{1}{2}[3 . (x-5)^{-1/2} + (-\tfrac{1}{2})(x-5)^{-3/2}(3x - 12)]$$

$$= \pm \tfrac{1}{2}(x-5)^{-3/2}[3(x-5) - \tfrac{1}{2}(3x-12)]$$

$$= \pm \tfrac{3}{4}(x-5)^{-3/2}[2x - 10 - (x-4)]$$

$$= \pm \tfrac{3}{4}(x-6)(x-5)^{-3/2} = \frac{\pm 3(x-6)}{4(x-5)^{3/2}}.$$

Putting $\dfrac{d^2y}{dx^2} = 0$, we get $x = 6$

Also $\dfrac{d^2y}{dx^2}$ changes sign at $x = 6$ *i.e.*, $\dfrac{d^3y}{dx^3} \neq 0$

$\therefore$ $x = 6$ determines the point of inflexion.

When $x = 6$ from (1), $y^2 = (6 - 2)^2 (6 - 5) = 16$ $\therefore$ $y = \pm 4$

Hence the points of inflexion are A(6, 4) and B(6, – 4).

**Double Points.** Writing (1) as,

$$f(x, y) = y^2 - (x-2)^2(x-5) = 0$$

For the double points, $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$

$$\therefore \quad \frac{\partial f}{\partial x} = -2(x-2)(x-5) - (x-2)^2 = 0$$

or $\quad (x-2)[2(x-5) + (x-2)] = 0$

or $\quad (x-2)(3x-12) = 0 \quad \therefore \quad x = 2, 4$

and $\quad \dfrac{\partial f}{\partial x} = 2y = 0 \quad$ or $\quad y = 0.$

$\therefore$ The possible double points are (2, 0) and (4, 0).

But only (2, 0) satisfies the given equation of the curve.

$\therefore$ The point P(2, 0) is the only double point.

Now we are to show that line joining the points of inflexion A (6, 4) and B(6, – 4) substends a right angle at the double point P(2, 0).

Now $m_1$ = slope of PA = $\left[\dfrac{4-0}{6-2}\right] = 1$ | *Using* $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

and $m_2$ = the slope of PB = $\dfrac{-4-0}{6-2} = -1$

$\because$ $m_1 \, m_2 = (1)(-1) = -1$

$\therefore$ the two line PA and PB are $\perp$ or the line AB subtends a right angle at P.

## KINDS OF CUSPS

We know that at a cusp, two branches of a curve have a common tangent and therefore a common normal.

### Single Cusp (Def.)

A cusp is said to be *single* if two branches of the curve lie entirely on **one side of the common normal.** [See. Figs. below]

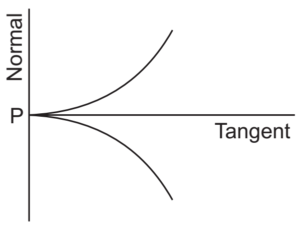



Fig. (*i*)

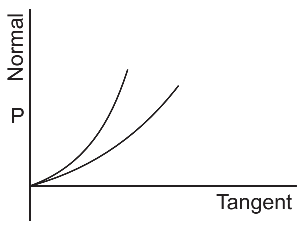



Fig. (*ii*)

### Double Cusp (Def.)

If the two branches of the curve extend to **both sides of the common normal** at the cusp (as shown in above Figs.), the cusp is called a **double cusp.**

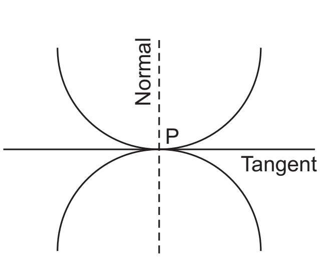



Fig. (*i*)

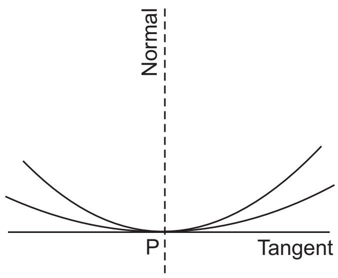



Fig. (*ii*)

## Cusp of the First Species

NOTES

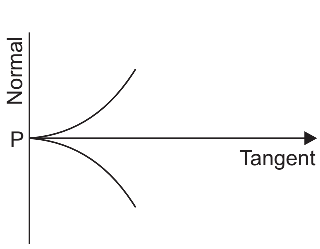



Fig. (*i*)

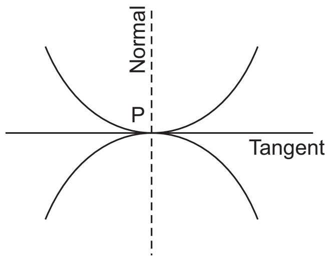



Fig. (*ii*)

**Def.** If the branches of the curve lie on **opposite sides of the common tangent** the cusp (as shown in the above Figs.) the cusp is called a **cusp of first species.**

## Cusp of the Second Species

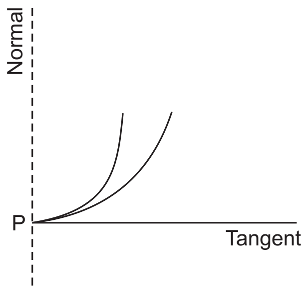



Fig. (*i*)

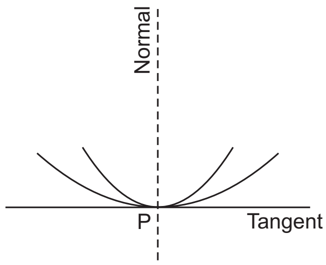
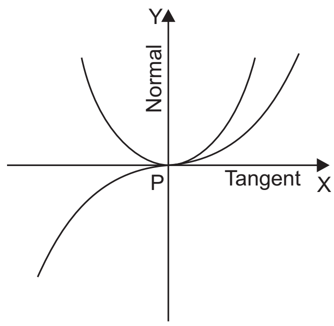



Fig. (*ii*)

**Def.** If the branches of the curve lie on the **same side of the common tangent** at the cusp, the cusp is called a cusp of second species, [See Figs. above]

## Point of Oscul-inflexion

A double cusp which is a combination of both the species is called a point of oscul-inflexion.




# WORKING RULE TO FIND THE NATURE OF CUSP AT THE ORIGIN

**Case I.** *When the cuspidal tangent are* $y^2 = 0$.

In this case we solve the given equation for $y$, neglecting terms containing powers of $y$ higher than two.

(*i*) If the roots are *real for only* **one** *sign of x,* the cusp is a **single cusp.**

(*ii*) If the roots are *real for* **both** *signs of x*, it is a **double cusp.**

(*iii*) If the roots are **opposite** *in sign*, the cusp is of the **first species.**

(*iv*) If the roots are of the **same sign,** the cusp is of the **second species.**

**Case II.** *When the cuspidal tangents are* $x^2 = 0$.

In this case, solve the given equation for $x$, neglecting powers of $x$ higher than two, and distinguish the various cases in a similar way as in case I.

**NOTES**

**Case III.** *When the cuspidal tangents are of the form*

$$(ax + by)^2 = 0.$$

Put $p = ax + by$. Then eliminate $y$ (or $x$, whichever is convenient) from this equation and the given equation of the curve. Suppose we eliminate $y$, then we shall get an equation in $p$ and $x$.

Solve the equation for $p$ (neglecting $p^3$ and higher powers of $p$) and discuss the various cases as in case I. (Here $p$ is $y$ of case I).

**Note.** Nature of the cusp at a point other than the origin.

Transfer the origin to that point and proceed as above.

## SOLVED EXAMPLES

**Example 17.** (*a*) *Show that curve* $(2a - x) = x$ *has a single cusp of the first species at the origin.*

(*b*) *Show that the curve* $y^3 = x^3 + ax^2$ *has a single cusp of first species at the origin.*

**Solution.** (*a*) The equation of the curve is

$$y^2(2a - x) = x^3 \qquad \text{...(1)}$$

Equating to zero the lowest degree terms, the tangents at the origin are given by $2ay^2 = 0$ or $y^2 = 0$, which are real and coincident.

Hence the origin is a cusp or a conjugate point.

From (1), $$y = \pm\, x\sqrt{\frac{x}{2a - x}} \qquad \text{...(2)}$$

When $x$ is small and + ve, $y$ is real $\therefore$ Real branches of the curve pass through the origin. Hence origin is a cusp.

Also for any small + ve value of $x$, the two values of $y$ are of **opposite** signs. $\therefore$ the cusp is a single cusp of first species.

(*b*) The equation of the curve is $y^3 = x^3 + ax^2$ ...(1)

Equating to zero the lowest degree terms, the tangents at the origin are given by $ax^2 = 0$ or $x^2 = 0$, which are real and coincident.

$\therefore$ Origin is a cusp or a conjugate points.

Solving (1) for $x$ (after neglecting $x^3$), we get

$$x = \pm\, y\sqrt{\frac{y}{a}} \qquad \text{...(2)}$$

For small + ve values of $y$, $x$ is real. $\therefore$ Real branches of the curve exist in the neighbourhood of origin. Hence origin is a cusp.

From (2), $x$ is real only if $y$ is + ve (*of one sign*), $\therefore$ the cusp is a single cusp.

Again for any small + ve value of $y$, the two values of $x$ are the opposite signs, the cusp is of first species.

Hence origin is a single cusp of first species.

NOTES

## EXERCISE–3

1. ($a$) Show that the curve $y^2 (x + 1) = x^4$ has a double cusp of the first species at the origin.

   ($b$) Find the nature of the cusp of the curve $y^2 = x^4(x + 2)$.

2. ($a$) Show that the curve $y^2 = (x - a)^2 (2x - a)$ has a single cusp of the first species at $(a, 0)$.

   ($b$) Show that the curve $(x + y)^3 - \sqrt{2} (y - x + 2)^2 = 0$ has a single cusp of the first species at the point $(1, - 1)$.

   ($c$) Show that the curve $x^2 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ has a single cusp of first species at $(- 1, - 2)$.

### Answers

1. ($b$) Double cusp of the first species.

# 13. CURVE TRACING

NOTES

**STRUCTURE**

Introduction
Procedure for Tracing Curves having Cartesian Equations
Polar Co-ordinates
Procedure for Tracing Curves having Polar Equations
Tracing of Parametric Equations
General Method to Find Asymptotes

## LEARNING OBJECTIVES

*After going through this unit you will be able to:*

- Polar Co-ordinates
- Procedure for Tracing Curves having Polar Equations
- Tracing of Parametric Equations
- General Method to Find Asymptotes

## INTRODUCTION

Let us consider the problem of tracing curves *i.e.,* of finding approximate shape of curves from their cartesian, polar or parametric equations without having to plot a large number of, points on them. We shall consider mainly those curves whose equations can be solved for $y$ $x$ or $r$. A working knowledge of the topics on Maxima, Minima ; concavity, points of Inflexion and asymptotes is essential for learning this chapter. In the following article, we shall explain the various steps which are helpful in tracing a curve. But these steps and (particularly their order) is by no means rigid and some steps can be omitted and their order can be varied to suit the particular problem.

## PROCEDURE FOR TRACING CURVES HAVING CARTESIAN EQUATIONS

**Symmetry**

(*i*) The curve is symmetrical about the **x-axis,** if the equation of the curve remains unchanged when **y is changed into – y** *i.e.,* if the equation of the curve contains only **even powers of y.**

NOTES

*e.g.*, $y^2 = x^2 + 4$ is symmetrical about $x$-axis where as $y^2 + yx = x^2$ is not symmetrical about $x$-axis.

(*ii*) The curve is symmetrical about the **y-axis** if the equation of the curve does not change when **x is changed into – x** *i.e.,* if the curve contains **only even powers of x.**

*e.g.*, $x^2 = 4ay$ is symmetrical about $y$-axis where as $y = x^3 - 3ax^2$ is not symmetrical about $y$-axis.

(*iii*) The curve is symmetrical about the **Line y = x** if the equation of the curve remains unchanged when **x is changed to y** and **y is changed to x.** ($y = x$ is a straight line through origin and making an angle of 45° with $x$-axis).

*e.g.*, $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$ where as $x^5 + y^5 = 5ax^2y$ is not symmetrical about the line $y = x$.

(*iv*) The curve is symmetrical about the **Line y = – x** if the equation of the curve remains unchanged when **x is changed to – y** and **y is changed to – x.** ($y = -x$ is a straight line ⊥ to $y = x$).

*e.g.*, $x^4 + y^4 = 4a^2xy$ is symmetrical about the line $y = -x$ where as $x^3 + y^3 = 3axy$ is not symmetrical about this line.

(*v*) The curve is symmetrical in **opposite quadrants** if the equation of the curve remains unchanged when **x is changed to – x** and **y is changed to – y.**

*e.g.*, $x^5 + y^5 = 5ax^2y$ is symmetrical in opposite quadrants whereas $x^3 + y^3 = 3axy$ is not symmetrical in opposite quadrants.

## Origin

(*a*) Find whether the curve passes through the origin. If the constant term is missing from the equation of the algebraic curve, then it passes through the origin.

If the algebraic curve passes through the origin, then write down the equation of the tangents at the origin **by equating the lowest degree terms to 0.**

If the origin is a *double point* (*i.e.,* there are two tangents at the origin) ; then find its nature whether a *node* (if the tangents are *real, distinct*) or a *cusp* (if the tangents are *real, coincident*) or a *conjugate* (or *isolated*) point if the tangents are *imaginary*. If a *cusp,* find its type.

A cusp is called a **single cusp** or a **double cusp** according as the two branches of the curve lie entirely *on one side* or *on both sides of the common normal.*

A cusp *single* or *double* is said to be of *first kind* or *second kind* according as the two branches of the curve , *lie on opposite or same side of the common tangent.*

(*b*) **Position of curve w.r.t. tangent at the origin**

Find $y_c - y_t$ (or $y_c^2 - y_t^2$ if square roots are there) where $y_c$ and $y_t$ are respectively the ordinates of a point on the curve and a point on the tangent at the origin for the same value of $x$.

Then discuss the two cases namely $x > 0$ and $x < 0$. (It should be noted that $x > 0$ in first and fourth quadrants and $x < 0$ in second and third quadrants).

If $(y_c - y_t)$ is + ve *i.e.,* $y_c > y_t$ ; then the curve lies above the tangent and if $(y_c - y_t)$ is – ve *i.e.,* $y_c < y_t$ ; then the curve lies below the tangent.

## Asymptotes

(*a*) *Working Rule for finding asymptotes of an algebraic curve.*

(*i*) **Asymptotes || to x-axis.** Equate to zero the co-efficient of the highest power of $x$, present in the given equation of the curve. Resolve L.H.S. into real linear factors.

(*ii*) **Asymptotes || to y-axis.** (Replace $x$ by $y$ in (*i*)).

(*iii*) **Oblique asymptotes**

Putting $x = 1$, $y = m$ in the highest degree terms of the given equation, find $\phi_n(m)$.

Putting $x = 1$, $y = m$ in the next lower degree terms of the given equation, find $\phi_{n-1}(m)$. Similarly, find $\phi_{n-2}(m)$.

Now put $\phi_n(m)$ equal to zero and solve for $m$. These are the slopes of the asymptotes.

For *distinct* real values of $m$, find $c$ from the equation

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}.$$

For two *equal* real values of $m$, find $c$ from the equation

$$\frac{c^2}{2!}\phi_n''(m) + c\phi'_{n-1}(m) + \phi_{n-2}(m) = 0.$$

Putting the values of $m$ and corresponding values of $c$ in $y = mx + c$, we get the required asymptotes.

(*b*) **Position of the curve w.r.t. asymptotes.**

(*i*) **Position of the curve w.r.t. asymptotes || to x-axis.**

*Let the asymptote parallel to x-axis be $y = a$.*

Find the value of $y - a$ from the equation of the curve.

(Generally, $(y - a)$ will occur as a factor in the equation of the curve.)

Now in the R.H.S. of this value of $y - a$, put $y = a$.

Then discuss the two cases when $x$ is $> 0$ (Near $\infty$) and $x < 0$ (Near $-\infty$).

If $(y - a)$ is + ve, the curve lies above the asymptote $y = a$ and if $(y - a)$ is – ve, the curve lies below the asymptote $y = a$.

(*ii*) **Position of the curve w.r.t. asymptotes parallel to y-axis.**

*Let the asymptote parallel to y-axis be $x = a$.*

Find the value of $x - a$ from the equation of the curve.

(Generally $(x - a)$ will occur as a factor in the equation of the curve).

Now in the R.H.S. of this value of $x - a$, put $x = a$.

Then discuss the two cases when $y > 0$ and $y < 0$.

(It may be noted that $y > 0$ in First and Second Quadrants and $y < 0$ in Third and Fourth Quadrants.)

If $(x - a)$ is + ve, the curve lies to the right of the asymptote $x = a$ and if $(x - a)$ is – ve, the curve lies to the left of the asymptote $x = a$.

(*iii*) **Position of the curve w.r.t. oblique asymptotes.**

Write the equation of the curve or one of its branches is in the form

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \ldots\ldots, \text{ then}$$

$y = mx + c$ is the asymptote to the curve.

$\therefore \quad y_c - y_a = \frac{A}{x} + \frac{B}{x^2} + \ldots\ldots$ where $y_c$ and $y_a$ stand for the ordinates of a point on the curve and a point on the asymptote (both having same abscissa).

Now discuss the two cases namely $x > 0$ and $x < 0$.

If $(y_c - y_a)$ is positive, curve lies above the asymptote.

If $(y_c - y_a)$ is negative, curve lies below the asymptote.

### Points of Intersections

(These can be obtained by solving the two equations).

Find intersections of the curve (*i*) with the $x$-axis ($y = 0$) ; (*ii*) with the $y$-axis ($x = 0$) (*iii*) with the line $y = x$ particularly if the curve is symmetrical about it.

(*iv*) With the line $y = -x$ particularly if the curve is symmetrical about it.

(*v*) With the asymptotes (if necessary).

(An asymptote of an $n$th degree curve cuts the curve in $(n - 2)$ points).

Also write down the equations of the tangents at these points of intersection by shifting the origin to the points.

### Region

Find regions in the four quadrants to which the curve is limited. This is usually done by solving for $y$ or for $x$ (or for each of the two) separately and considering both positive and negative values of $x$ (or $y$). Values of $x$ (or $y$) which make $y$ (or $x$) imaginary are to be rejected.

Again, such values* which make the left and right members of an equation opposite in sign are to be rejected.

(For example, see the solved examples).

**Note.** If $x^2 \le a^2$ ; then $-a \le x \le a$.

If $x^2 \ge a^2$ ; then either $x \ge a$ or $x \le -a$.

If $(x - \alpha)(x - \beta) \le 0$ ; then $x$ lies between $\alpha$ and $\beta$.

If $(x - \alpha)(x - \beta) \ge 0$ ; then $x$ does not lies between $\alpha$ and $\beta$.

### Special points (or singularities)

(*a*) **Find** $\mathbf{\dfrac{dy}{dx}}$

$\left(\dfrac{dy}{dx}\right.$ represents the slope of the tangent to the curve at the point $\left.(x, y)\right)$

Find the points where

(*i*) $\dfrac{dy}{dx} = 0$ *i.e.,* tangents are $\parallel$ to $x$-axis

(*ii*) $\dfrac{dy}{dx} = \infty$ *i.e.,* tangent are $\parallel$ to $y$-axis

(*iii*) $\dfrac{dy}{dx}$ is positive *i.e.,* the function is increasing *i.e.,* the curve is rising.

(*iv*) $\dfrac{dy}{dx}$ is negative *i.e.,* the curve is falling.

**Find** $\mathbf{\dfrac{d^2y}{dx^2}}$ (If not tedious)

Find the points where

(*i*) $\dfrac{d^2y}{dx^2}$ is + ve *i.e.,* the curve is concave upwards.

---

* In Quadrant I, $x$ is + ve and $y$ is + ve.
In Quadrant II, $x$ is – ve and $y$ is + ve.
In Quadrant III, $x$ is – ve and $y$ is – ve.
In Quadrant IV, $x$ is + ve and $y$ is – ve.

(*ii*) $\frac{d^2y}{dx^2}$ is – ve *i.e.,* the curve is concave downwards.

(*iii*) Also find **points of Inflexion.**

(At a point of Inflexion, curve crosses the tangent. At a point of inflexion $\frac{d^2y}{dx^2}$ $= 0$ and $\frac{d^3y}{dx^3} \neq 0$ or $\frac{d^2y}{dx^2}$ changes sign while passing through it).

**Remark.** The steps of Art 2. can be remembered as :

**R—SOAP**

where S ≡ Symmetry, O ≡ origin, A ≡ Asymptotes ; P ≡ Points

(*i*) of intersection with $y = 0$, $x = 0$, $y = x$, $y = -x$

(*ii*) points where $\frac{dy}{dx} = 0$ or $\infty$ or $> 0$ or $< 0$,

and R ≡ Region.

**Note.** Sometimes inconvenient steps may be omitted without any disadvantage.

## SOLVED EXAMPLES

**Example 1.** *Trace the curve* $9ay^2 = (x - 2a)(x - 5a)^2$.

**Sol.** The equation of the curve is $9ay^2 = (x - 2a)(x - 5a)^2$ ...(1)

1. **Symmetry.** Since (1) contains only even powers of $y$.

∴ The curve is symmetrical about $x$-axis.

2. **Origin.** The curve does not pass through the origin because (0, 0) does not satisfy the given equation.

3. **Asymptotes.** The curve has no asymptotes.

4. **Points of Intersection**

(*i*) **Intersection with x-axis**

Putting $y = 0$ in (1), we get $(x - 2a)(x - 5a)^2 = 0$ or $x = 2a, 5a$.

Thus, the curve meets $x$-axis in the points A($2a$, 0) and B($5a$, 0).

Shifting the origin to ($2a$, 0) by putting $x = X + 2a$, $y = Y + 0$ ; the equation (1) transforms to $9aY^2 = X(X - 3a)^2$.

Equating to zero the lowest degree terms, the tangent at the new origin is $X = 0$ *i.e.,* new $y$-axis.

Thus, the tangent at A($2a$, 0) is parallel to old $y$-axis.

Shifting the origin to B($5a$, 0) by putting $x = X + 5a$ and $y = Y + 0$, the equation (1) transforms to $9aY^2 = (X + 3a)X^2$

∴ Tangents at new origin are $9aY^2 = 3aX^2$ *i.e.,* $Y^2 = \frac{X^2}{3}$ ∴ $Y = \pm\frac{X}{\sqrt{3}}$

∴ New origin is a Node.

(*ii*) **Intersections with y-axis**

Putting $x = 0$ in (1), we get $9ay^2 = -50a^3$, which gives imaginary values of $y$. Therefore, the curve does not meet $y$-axis.

NOTES

5. **Region**

From (1), $$y = \frac{1}{\sqrt{9a}}(x - 5a)\sqrt{x - 2a} \qquad \text{...(2)}$$

(Taking only positive sign with square root)

(This value of $y$, corresponds to the part of the curve in first two quadrants.)

When $x < 2a$, $y$ is imaginary.

Thus, the curve does not lie to the left of the line $x = 2a$.

When $x > 2a$, $y$ is real.

6. **Special points**

From (2), $$\frac{dy}{dx} = \frac{1}{\sqrt{9a}}\left[(x - 5a)\frac{1}{2\sqrt{x - 2a}} + \sqrt{x - 2a}\right]$$

or $$\frac{dy}{dx} = \frac{1}{\sqrt{9a}}\left[\frac{x - 5a + 2(x - 2a)}{2\sqrt{x - 2a}}\right] = \frac{3(x - 3a)}{2.3\sqrt{a}\,.\sqrt{x - 2a}}.$$

or $$\frac{dy}{dx} = \frac{x - 3a}{2\sqrt{a}\sqrt{x - 2a}} \qquad \text{...(3)}$$

$\frac{dy}{dx} = 0$ gives $x = 3a$.

But when $x = 3a$, then from (1) $9ay^2 = 4a^3$.

$\therefore \quad y = \pm\frac{2a}{3}$

$\therefore$ Tangents at the points $\left(3a, \pm\frac{2a}{3}\right)$ are parallel to $x$-axis.

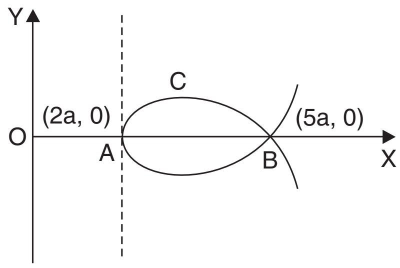
When $x > 3a$, $\frac{dy}{dx}$ is positive | From (3)

$\therefore$ The curve rises for all values of $x > 3a$ (provided $y$ is + ve) when $x < 3a$, $\frac{dy}{dx}$ is negative.

$\therefore$ The curve is falling for values of $x < 3a$.

Again $\frac{dy}{dx} \to \infty$ when $x = 2a$.

$\therefore$ The tangent at $(2a, 0)$ is || to $y$-axis. Thus, the shape of the curve is as shown in the figure.




**Note. Two important results from real analysis :**

1. $\mathbf{x^2 < a^2 \Rightarrow -a < x < a.}$ 2. $\mathbf{x^2 > a^2 \Rightarrow x > a \text{ or } x < -a.}$

**Example 2.** *Trace the curve* $y^2(2a - x) = x^3$.

**Sol.** The equation of the curve is $y^2(2a - x) = x^3$ ...(1)

1. **Symmetry.** Since (1) contains only even powers of $y$, the curve is symmetrical about $x$-axis.

2. **Origin.** The tangents at the origin are given by $y^2 = 0$ *i.e.*, $y = 0$, $y = 0$. Since the two tangents are real and coincident, $\therefore$ origin is a cusp.

3. **Asymptotes.** Equating to zero, the co-eff. of $y^2$, the highest degree term in $y$, the asymptote parallel to $y$-axis is $x - 2a = 0$ *i.e.*, $x = 2a$. There is no other asymptote of the curve.

We can discuss the position of the curve w.r.t. asymptote $x = 2a$.

NOTES

4. **Points of intersections.** The curve meets $x$-axis and $y$-axis at the origin only.

5. **Region.** From (1), $y = x\sqrt{\dfrac{x}{2a - x}}$

when $x < 0$, $y$ is imaginary.

$\therefore$ No portion of the curve lies to the left of the line $x = 0$ *i.e.,* $y$-axis.

When $0 < x < 2a$, $y$ is Real.

When $x > 2a$, $y$ is Imaginary.

$\therefore$ No portion of the curve lies to the right of the line $x = 2a$.

6. **Special points**

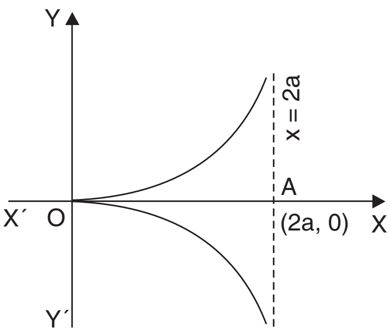
From (1), $$y = \frac{x^{3/2}}{\sqrt{2a - x}} \qquad \ldots(2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{x}\,(3a - x)}{(2a - x)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx} = 0$$

when $\sqrt{x}\,(3a - x) = 0$

or $x = 0, \quad x = 3a$




Rejecting $x = 3a$, because when $x = 3a$, from (2) $y$ is imaginary.

When $x = 0$, $y = 0$ $\therefore$ Tangent at (0, 0) *i.e.,* at origin is || to $x$-axis *i.e.,* the tangent at origin is $x$-axis (Also see step 2).

Again, $\dfrac{dy}{dx} \to \infty$ when $x \to 2a$. From (2), when $x \to 2a$, $y \to \infty$

Thus, $x = 2a$ is an asymptote (Also see step 3)

when $0 < x < 2a$, $\dfrac{dy}{dx}$ is positive.

$\therefore$ For positive values of $y$, $y$ is an increasing function of $x$ *i.e.,* the curve rises for values of $x$ between 0 and $2a$.

Thus, the shape of curve is as shown in the figure.

**Example 3.** *Trace the curve* $y = x^3 - 3ax^2$.

**Sol.** The curve is $y = x^3 - 3ax^2$ ...(1) or $y = x^2(x - 3a)$

1. **Symmetry.** No symmetry.

2. **Origin.** The curve passes through the origin.

The equation of tangent at the origin is $y = 0$ *i.e.,* $x$-axis.

3. **Asymptotes.** No asymptotes.

4. **Points of intersection**

*Intersections with x-axis.* Putting $y = 0$ in (1), we get

$$x^3 - 3ax^2 = 0 \quad \text{or} \quad x^2(x - 3a) = 0$$

$$\therefore \quad x = 0 \quad \text{or} \quad x = 3a$$

$\therefore$ Intersections with $x$-axis are (0, 0) and $(3a, 0)$

Shifting the origin to $(3a, 0)$, (1) becomes

$$Y = (X + 3a)^2X$$

$\therefore$ Equation of the tangent at the new origin is $Y = 9a^2X$.

5. **Region**

From (1), $y = x^2(x - 3a)$

when $x < 0$, $y$ is – ve.

$\therefore$ No portion of the curve lies in the second quadrant.

When $0 < x < 3a$, $y$ is – ve.

$\therefore$ The curve lies in fourth quadrant for values of $x$ such that $0 < x < 3a$.

When $x > 3a$, $y$ is + ve.

$\therefore$ The curve lies in the first quadrant for values of $x > 3a$.

6. **Special points**

From (1), $$\frac{dy}{dx} = 3x^2 - 6ax = 3x(x - 2a) \qquad ...(2)$$

$$\frac{dy}{dx} = 0 \text{ gives } x = 0 \text{ and } x = 2a.$$

When $x = 0$, then from (1) $y = 0$

$\therefore$ Tangent at (0, 0) is $x$-axis.

when $x = 2a$, from (1), $y = 8a^3 - 12a^3 = -4a^3$

$\therefore$ Tangent at $(2a, -4a^3)$ is parallel to $x$-axis.

When $x < 0$, then from (2), $\frac{dy}{dx}$ is + ve.

$\therefore$ $y$ increases as $x$ increases *i.e.,* the curve is rising.

When $0 < x < 2a$, $\frac{dy}{dx}$ is – ve.

$\therefore$ The curve falls in this portion.

When $x > 2a$, $\frac{dy}{dx}$ is + ve.

$\therefore$ Again $y$ increases as $x$ increases. ($\forall\, x > 2a$)

From (2), $$\frac{d^2y}{dx^2} = 6x - 6a = 6(x - a)$$

$$\frac{d^2y}{dx^2} = 0 \text{ gives } x = a$$

$$\frac{d^3y}{dx^3} = 6.$$

At $x = a$, $$\frac{d^3y}{dx^3} = 6 \neq 0$$

$\therefore$ $x = a$ gives a point of inflexion.

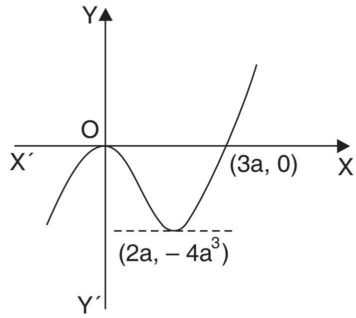
When $x = a$, then from (1)

$$y = x^3 - 3ax^2 = a^3 - 3a^3 = -2a^3$$

$\therefore$ $(a, -2a^3)$ gives a point of inflexion.

When $x > a$, $\frac{d^2y}{dx^2}$ is positive.

$\therefore$ The curve is concave upwards for values of $x > a$.




When $x < a$, $\frac{d^2y}{dx^2}$ is negative.

$\therefore$ The curve is concave downwards for values of $x < a$.

The shape of the curve is as shown in the figure.

**Example 4.** *Trace the curve* $x^{2/3} + y^{2/3} = a^{2/3}$ (**Astroid**).

**Sol.** The curve is $x^{2/3} + y^{2/3} = a^{2/3}$ ...(1)

1. **Symmetry.**

(*i*) The curve is symmetrical about $x$-axis.

($\because$ on changing $y$ to $-y$, the equation remains unchanged)

(*ii*) The curve is symmetrical about $y$-axis.

(*iii*) The curve is symmetrical about the line $y = x$.

(*iv*) The curve is symmetrical about the line $y = -x$.

(*v*) The curve is also symmetrical in opposite quadrants.

2. **Origin**

The curve does not pass through the origin.

3. **Asymptotes**

The curve has no asymptotes.

4. **Points of Intersection**

(*i*) *To find intersections with x-axis* (Putting $y = 0$ in (1), we have)

$$x^{2/3} = a^{2/3}$$

cubing, $x^2 = a^2 \quad \therefore \quad x = \pm a.$

$\therefore$ Intersections with $x$-axis are $(a, 0)$ and $(-a, 0)$.

(*ii*) Similarly, intersections with $y$-axis are $(0, a)$ and $(0, -a)$.

(*iii*) *To find intersections with line* $y = x$.

Putting $y = x$ in (1), $x^{2/3} + x^{2/3} = a^{2/3}$ or $2x^{2/3} = a^{2/3}$

$\therefore$ Cubing, $8x^2 = a^2 \quad \therefore \quad x = \pm \dfrac{a}{2\sqrt{2}}$.

$\therefore$ Intersections are $\left(\pm \dfrac{a}{2\sqrt{2}}, \pm \dfrac{a}{2\sqrt{2}}\right)$.

(*iv*) Similarly, intersections with the line $y = -x$ are $\left(\pm \dfrac{a}{2\sqrt{2}}, \mp \dfrac{a}{2\sqrt{2}}\right)$.

5. **Region**

From (1), $y^{2/3} = a^{2/3} - x^{2/3} \quad \therefore \quad y = ((a^{2/3} - x^{2/3})^3)^{1/2}$

For $y$ to be real, $a^{2/3} - x^{2/3} > 0$ or $a^{2/3} > x^{2/3}$.

*i.e.,* $x^{2/3} < a^{2/3} \quad \therefore \quad x^2 < a^2 \quad \therefore \quad -a < x < a.$

$\therefore$ The curve lies between the lines $x = -a$ and $x = a$.

Interchanging the roles of $x$ and $y$, we can prove that the curve lies between the lines $y = -a$ and $y = a$.

6. **Special points**

Differentiating (1), $\dfrac{2}{3} x^{-1/3} + \dfrac{2}{3} y^{-1/3} \dfrac{dy}{dx} = 0 \quad \therefore \quad \dfrac{dy}{dx} = -\dfrac{x^{-1/3}}{y^{-1/3}} = -\dfrac{y^{1/3}}{x^{1/3}}$ ...(2)

If $\dfrac{dy}{dx} = 0$, then $y^{1/3} = 0$ or $y = 0$

putting $y = 0$ in (1), $x = \pm a$.

$\therefore$ Tangents are || to $x$-axis at the points $(\pm a, 0)$ *i.e.,* $x$-axis is a tangent at the points $(\pm a, 0)$.

Similarly, $\dfrac{dy}{dx} \to \infty$ when $x = 0$

putting $x = 0$ in (1), $y = \pm a$.

NOTES

$\therefore$ Tangents are || to $y$-axis at the points $(0, \pm a)$.

Again from (2), $\frac{dy}{dx}$ is negative when $x$ and $y$ are both positive.

$\therefore$ $y$ decreases as $x$ increases. (In the first quadrant)

Diff. (2), $$\frac{d^2y}{dx^2} = -\left[\frac{x^{1/3}\cdot\frac{1}{3}y^{-2/3}\frac{dy}{dx} - y^{1/3}\cdot\frac{1}{3}x^{-2/3}}{x^{2/3}}\right]$$

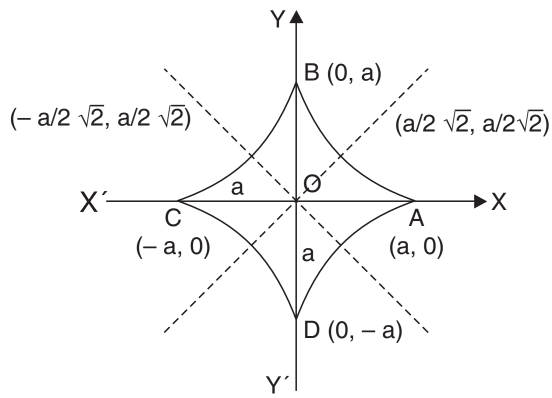
or $$\frac{d^2y}{dx^2} = -\frac{1}{3}\left[\frac{\frac{x^{1/3}}{y^{2/3}}\left(-\frac{y^{1/3}}{x^{1/3}}\right) - \frac{y^{1/3}}{x^{2/3}}}{x^{2/3}}\right] = \frac{1}{3}\left[\frac{x^{2/3}+y^{2/3}}{x^{4/3}y^{1/3}}\right]$$

or $$\frac{d^2y}{dx^2} = \frac{a^{2/3}}{3x^{4/3}\,y^{1/3}}$$ [By (1)]

$\frac{d^2y}{dx^2}$ is positive when $x$ and $y$ are both positive.

$\therefore$ Curve is concave upwards in first quadrant.

The shape of the curve is as shown in the figure.




**Note.** The point where the line $y = x$ meets the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ $\left(i.e., \text{ the point } \left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)\right)$ is called the **VERTEX** of the astroid.

## EXERCISE A

*Trace the following curves :*

**1.** $y^2 = x^3$.

**2.** $9ay^2 = x(x - 3a)^2$.

**3.** $a^2y^2 = x^3(2a - x)$.

**4.** $a^4y^2 = a^2x^4 - x^6$.

**5.** $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ or $x^2(x^2 + y^2) = a^2(x^2 - y^2)$.

**6.** $a^2y^2 = x^2(a^2 - x^2)$.

**7.** $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$ or $x^2(y^2 + a^2) + y^2(y^2 - a^2) = 0$.

**8.** $y^2 = (x - 2)^2(x - 5)$.

**9.** $a^2x^2 = y^3(2a - y)$

**10.** $y^2 = (x - 1)(x - 2)(x - 3)$.

**11.** $xy^2 = 4a^2(2a - x)$.

**12.** (*a*) $x^2y^2 = (a + y)^2(a^2 - y^2)$ (*b*) $x^2y^2 = (a^2 + y^2)(a^2 - y^2)$.

**13.** $y(x^2 + 4a^2) = 8a^3$.

**14.** $y^2x = a^2(x - a)$.

**15.** $y^2(x^2 - 1) = x$.

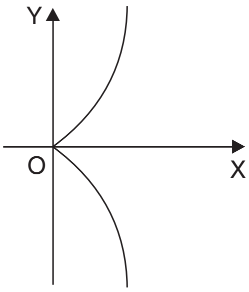
**16.** (*a*) $y = \frac{x^2+1}{x^2-1}$ (*b*) $x^2y^2 = x^2 + 1$.

**17.** $x^2y^2 = a^2(y^2 - x^2)$.

## Answers

**NOTES**

1.

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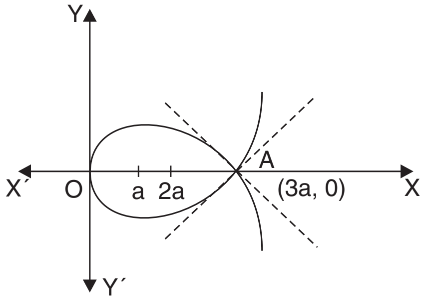


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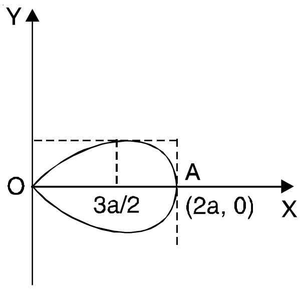


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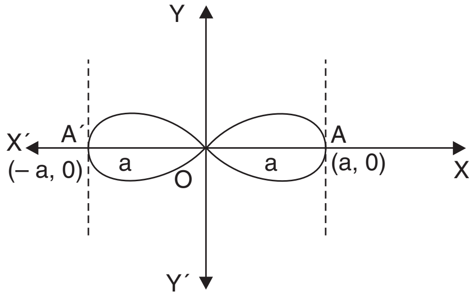


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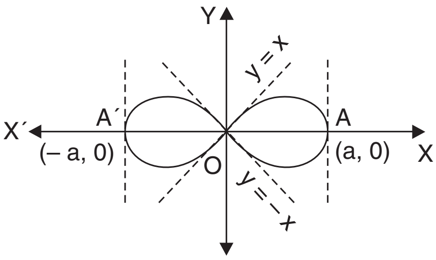


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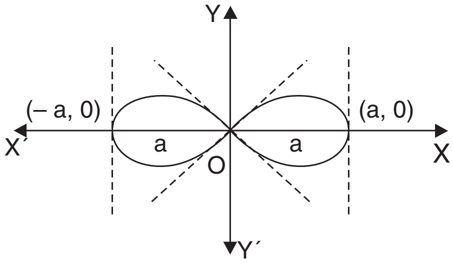


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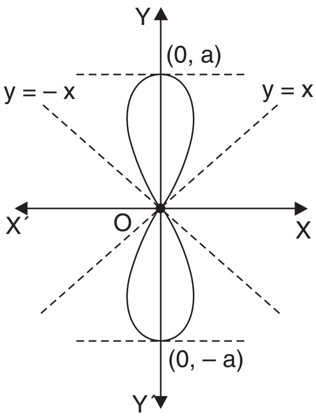


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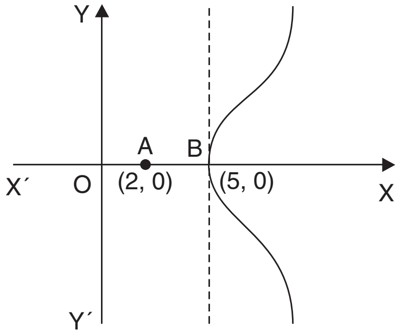


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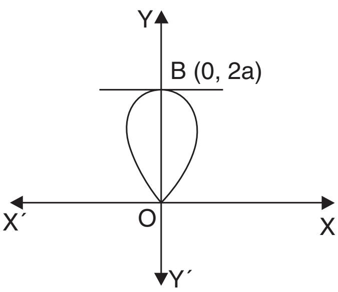


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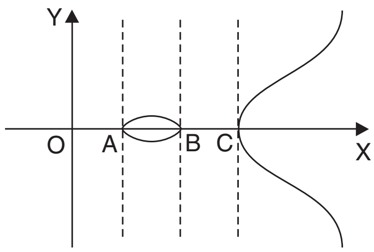
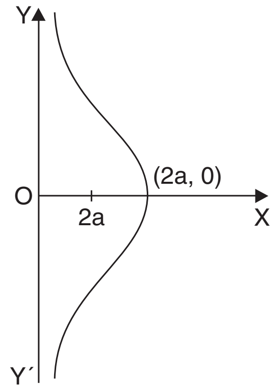
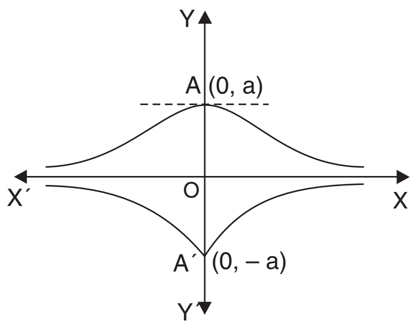


**NOTES**

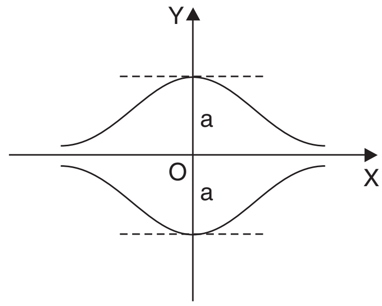
**11.**



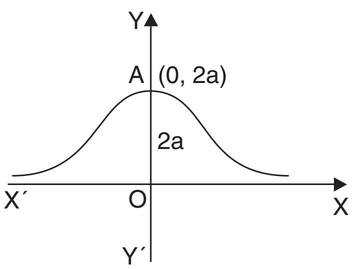

**12.** (*a*)



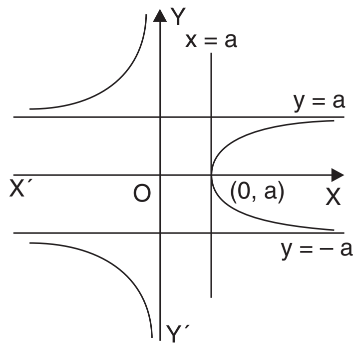

(*b*)



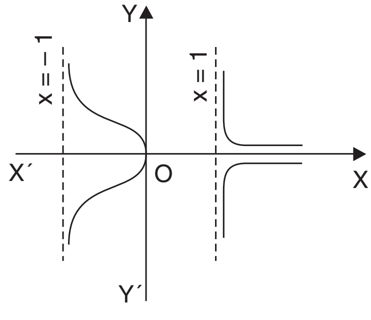

**13.**




**14.**

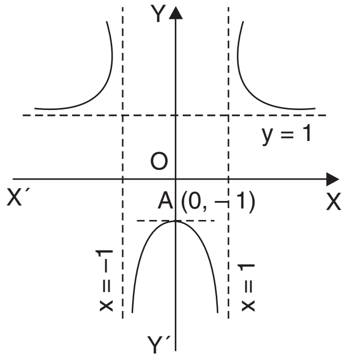



**15.**

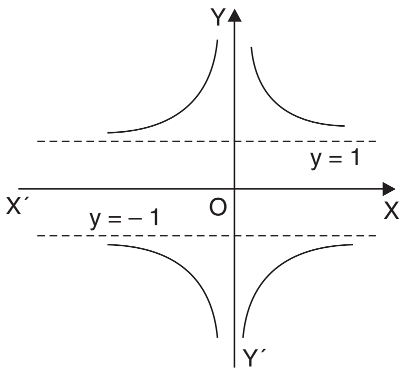


NOTES

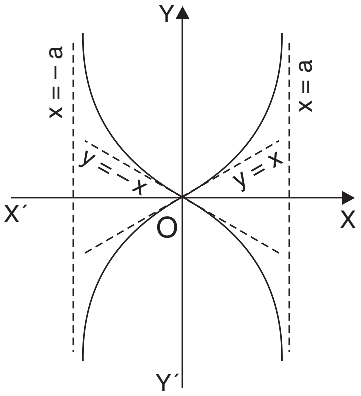
**16.** (*a*)



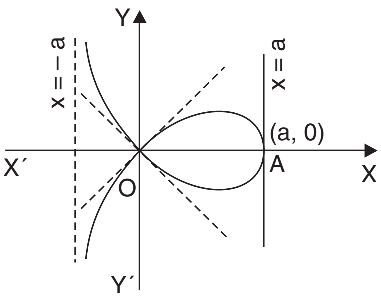

(*b*)




**17.**

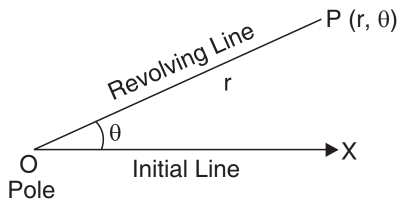



**18.**




## POLAR CO-ORDINATES

If we have any horizontal line OX called the *initial line* and another line called the revolving line, makes an angle θ with the initial line, then the polar co-ordinates of a point P on it where OP = $r$ arc ($r$, θ).




The point O is called the *pole* and the angle θ is called the *vectorial angle* of the point P and the length $r$ is called its *radius vector*.

**Note 1. Representation of (–r, θ) when r > 0.**

*Polar radius r is considered positive if it is measured from the pole along the half ray bounding the vectorial angle.*

*Let P be the point* ($r$, θ). *Produce OP backwards and cut off OP′ = OP.*

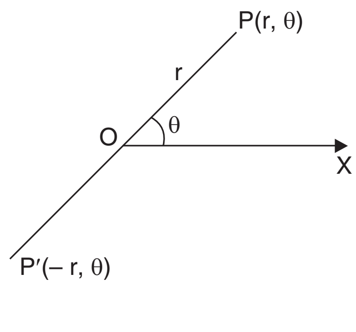
*Then P′* is (– $r$, θ).

**Remark.** For $r \geq 0$, (– $r$, θ) is the same point as ($r$, θ + π) or ($r$, θ) is the same point as (– $r$, θ + π).

**Note 2.** In polar co-ordinates θ = α is the equation of the *half ray* OP passing through the pole and making an angle α with the initial line.

The equation of the other half ray OP′ is θ = π + α.

**Note 3.** $r = a$ is the equation of a circle having centre at the pole and radius $a$.



NOTES

**Note 4. Relations between cartesian and polar co-ordinates,**

$$\mathbf{x = r \cos \theta, \; y = r \sin \theta}$$

Squaring and adding, $r = \sqrt{x^2 + y^2}$.

Dividing the two, $\tan \theta = \frac{y}{x}$.

## PROCEDURE FOR TRACING CURVES HAVING POLAR EQUATIONS

### Symmetry

**A curve is symmetrical about the ray $\theta = \alpha$ if on changing $\theta$ to $2\alpha - \theta$, the equation remains unchanged.**

In particular :

(*i*) The curve is symmetrical about the initial line $\theta = 0$ (*i.e.,* $x$-axis) if the equation remains unchanged on changing $\theta$ to $-\theta$. ($\because \quad 2\alpha - \theta = 0 - \theta = -\theta$)

For example, $r = a(1 + \cos \theta)$, $r = a \cos 3\theta$ are symmetrical about the initial line.

(*ii*) The curve is symmetrical about the half ray $\theta = \frac{\pi}{2}$.

(*i.e.,* $y$-axis) if the equation remains unchanged on changing $\theta$ to $(\pi - \theta)$

[$\because \quad 2\alpha - \theta = \pi - \theta$] *e.g.,* $r = a(1 + \sin \theta)$ and $r = a \sin 3\theta$ are symmetrical about $\theta = \frac{\pi}{2}$.

If the equation of the curve remains unchanged *when $\theta$ is changed to $-\theta$ and $r$ to $-r$,* even then the curve is symmetrical about the half ray $\theta = \frac{\pi}{2}$.

For example, $r\theta = a$ is symmetrical about $\theta = \frac{\pi}{2}$.

(*iii*) The curve is symmetrical about the **pole** (*i.e.,* symmetrical in opposite quadrants ) if the equation remains unchanged on changing $r$ to $-r$ **or** $\theta$ to $\pi + \theta$.

### Pole or origin

(*i*) *Find whether the curve passes through the pole or not.* This can be done by putting $r = 0$ in the equation and if on doing so, we get some real value (or values) of $\theta$ ; then the curve passes through the pole.

If it is not possible to find real value of $\theta$ for which $r = 0$, the curve does not pass through the pole.

(*iii*) *Find the tangents at the pole.* **Putting r = 0, the real values of $\theta$ give the tangents at the pole.**

For example, consider the curve $r = a(1 - \cos \theta)$. Putting $r = 0$, we get

$$\cos \theta = 1 \; \therefore \quad \theta = 0.$$

Hence the curve passes through the pole and the line $\theta = 0$ *i.e.,* the initial line is the tangent at the pole.

Again, consider $r = a(3 - \sin \theta)$.

Putting $r = 0$, we get $\sin \theta = 3$ which does not give any real value of $\theta$

$[\because \quad |\sin \theta| \le 1]$.

Hence the curve does not pass through the pole.

(*iii*) Find the points where the curve cuts the initial line and the line $\theta = \pi/2$.

**Asymptotes**

(*i*) If $\theta \to \theta_1$ (some finite value) when $r \to \infty$, then there is an asymptote.

**Working Rule to find polar asymptotes :**

Put $r = \dfrac{1}{u}$ and let $u \to 0$ so that $\theta \to \theta_1$ ($\theta_1$ is a definite number)

Determine $$p = \underset{\substack{\theta \to \theta_1 \\ u \to 0}}{\text{Lt}} \left(-\frac{d\theta}{du}\right).$$

Putting the values of $p$ and $\theta_1$ in the equation $p = r \sin (\theta_1 - \theta)$, we get the corresponding asymptote.

(*ii*) If as $\theta \to \infty$, $r \to a$, then there is a circular asymptote $r = a$.

Circular asymptotes of the curve $r = f(\theta)$ are given by $r = \underset{\theta \to \infty}{\text{Lt}} f(\theta)$.

4. **Points of Intersection.** Find some points on the curve for convenient values of $\theta$. (especially for values of $\theta = \alpha$ for which the curve is symmetrical).

5. **Region.** Solve the given equation for $r$ or $\theta$ (if possible). Find the regions in which the curve does not lie. This can be done in the following way :

(*i*) If for $\theta_1 < \theta < \theta_2$, $r$ is imaginary, then there is no portion of the curve between the lines $\theta = \theta_1$ and $\theta = \theta_2$. Consider $r^2 = a^2 \cos 2\theta$.

For $\pi/4 < \theta < 3\pi/4$, $\cos 2\theta$ is negative. $\therefore$ $r^2$ is –ve and so $r$ is imaginary. Hence the curve does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

(*ii*) If the greatest and least numerical values of $r$ be respectively $a$ and $b$, the curve lies entirely within the circle $r = a$ and entirely outside the circle $r = b$.

(*iii*) Trace the variations of $r$ when $\theta$ varies in the intervals $(0, \infty)$ and $(-\infty, 0)$ marking the values of $\theta$ for which $r = 0$ or attains maximum and minimum values. Plot the points so obtained.

[**Note.** When $r$ is a periodic function of $\theta$, the negative values of $\theta$ need not be considered. We may consider values from $\theta = 0$ to those values of $\theta$ where the values begin to repeat.]

6. **Value of** $\phi$. Find $\tan \phi = r\dfrac{d\theta}{dr}$ and $\phi$.

($\phi$ is the angle between the tangent and radius vector.)

Find the points where $\phi$ is 0 or $\dfrac{\pi}{2}$.

Again, $r$ increases as $\theta$ increases if $\dfrac{dr}{d\theta}$ as positive and $r$ decreases as $\theta$ increases if $\dfrac{dr}{d\theta}$ is negative.

**Note. Conversion into cartesians.** Transform the equation into cartesians, when tracing of the curve becomes easy on transformation to cartesian system of coordinate.

## SOLVED EXAMPLES

NOTES

**Example 5.** *Trace the curve* $r = a(1 - \cos \theta)$.

**Sol.** The equation of the curve is $r = a(1 - \cos \theta)$ ...(1)

1. **Symmetry.** The curve is symmetrical about the initial line because the equation of curve remains unaltered when $\theta$ is changed to $-\theta$.

2. **Pole or origin.** (*i*) When $\theta = 0$, $r = 0$ hence the curve passes through the pole and the tangent at the pole is the line $\theta = 0$, *i.e.,* the initial line.

(*ii*) The curve meets the initial line $\theta = 0$ at (0, 0) and the lines $\theta = \pi/2$ and $\pi$ in the points $(a, \pi/2)$ and $(2a, \pi)$ respectively.

3. **Asymptotes.** Since for any finite value of $\theta$, $r$ does not tend to infinity, $\therefore$ the curve has no asymptote.

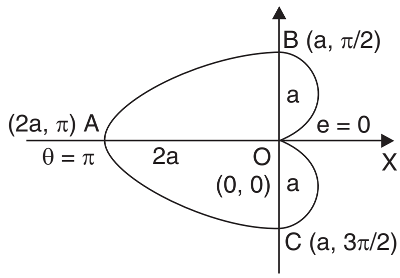
4. **Point of Intersection.** The corresponding values of $\theta$ and $r$ are given below :

| | | | | |
|---|---|---|---|---|
| $\theta = 0$ | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ |
| $r = 0$ | $a/2$ | $a$ | $3a/2$ | $2a$. |

[We need not trace the curve for values of $\theta$ from $\pi$ to $2\pi$ as the curve is symmetrical about the initial line.]

5. **Region.** (*i*) $\because$ $|\cos \theta| \le 1$ $\therefore$ From (1), $r \le 2a$.

$\therefore$ The curve lies entirely within the circle $r = 2a$.




(*ii*) When $\theta$ increases from 0 to $\pi$, $r$ remains positive and increases from 0 to $2a$.

When $\theta$ increases from $\pi$ to $2\pi$, $r$ is positive and decreases from $2a$ to 0.

6. **Value of** $\phi$.

From (1), $$\frac{dr}{d\theta} = a \sin \theta.$$

$$\therefore \quad \tan \phi = r\frac{d\theta}{dr} = \frac{r}{\left(\dfrac{dr}{d\theta}\right)} = \frac{a(1-\cos\theta)}{a \sin \theta} = \frac{a \,.\, 2\sin^2 \dfrac{\theta}{2}}{a \,.\, 2 \sin \dfrac{\theta}{2} \cos \dfrac{\theta}{2}} = \tan \frac{\theta}{2}.$$

$$\therefore \quad \phi = \frac{\theta}{2}.$$

Now $\phi = 0$ when $\theta = 0$ and $\phi = \pi/2$ when $\theta = \pi$.

$\therefore$ At $(2a, \pi)$ the tangent is perpendicular to the line $\theta = \pi$.

The shape of the curve is as shown in the figure.

**Note.** We will not consider the values of $\theta$ after $2\pi$ here, because $r$ is periodic with period $2\pi$ and therefore we do not get any new point on the curve.

**Example 6.** *Trace the curve* $r = a \sin 3\theta$.

**Sol.** The equation of the curve is $r = a \sin 3\theta$ ...(1)

1. **Symmetry**

From (1), $$r = a \sin 3\theta = a \sin (\pi - 3\theta) = a \sin (3\pi - 3\theta) \text{ etc.}$$

or $$r = a \sin 3\theta = a \sin 3\left(\frac{\pi}{3} - \theta\right) = a \sin 3 (\pi - \theta) \text{ etc.}$$

NOTES

$\therefore$ (1) remains unchanged when θ is changed to $\frac{\pi}{3} - \theta$, or $\pi - \theta$.

$\therefore$ The curve is symmetrical about the rays

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}\left(= \frac{3\pi}{6}\right), \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$$

($\because$ If on changing θ to $2\alpha - \theta$, the equation is unchanged, symmetry is about the ray $\theta = \alpha$).

2. **Pole or origin.** Putting $r = 0$, we get $\sin 3\theta = 0$

$\therefore$ $3\theta = n\pi$ where $n$ is zero or any other integer.

$\therefore$ $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{6\pi}{3}$

$\therefore$ The curve passes through the pole and the tangents at the pole are

$$\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \theta = \frac{3\pi}{3}, \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3}, \theta = \frac{6\pi}{3}.$$

3. **Asymptotes.** Putting $r = \frac{1}{u}$; from (1), $u = \frac{1}{a \sin 3\theta}$

$u = 0$ does not give any finite value of θ.

$\therefore$ No asymptotes.

4. **Point of Intersection**

When $\theta = 0, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$ (Rays of symmetry)

From (1), $r = 0, a, -a, a, -a, a, -a.$

5. **Region.** (*i*) From (1), $r \le a$ numerically [$\because$ $\sin 3\theta \le 1$ numerically]

$\therefore$ The curve lies entirely within the circle $r = a$.

(*ii*) *Region w.r.t. tangents at the pole.* Tangents at the pole are

$$\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \theta = \frac{3\pi}{3}, \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3}$$

When $0 \le \theta \le \pi/3$ ; then $0 \le 3\theta \le \pi$ *i.e.,* $3\theta$ lies in first or second quadrant.

$\therefore$ From (1), $r$ is + ve [$\because$ $\sin \theta$ is + ve in I or II quadrant]

Similarly, when $\frac{\pi}{3} \le \theta \le \frac{2\pi}{3}$, $r$ is – ve.

When $\frac{2\pi}{3} \le \theta \le \frac{3\pi}{3}$, $r$ is + ve.

When $\frac{3\pi}{3} \le \theta \le \frac{4\pi}{3}$, $r$ is – ve.

When $\frac{4\pi}{3} \le \theta \le \frac{5\pi}{3}$, $r$ is + ve.

When $\frac{5\pi}{3} \le \theta \le \frac{6\pi}{3}$, $r$ is – ve.

6. **Value of φ.** From (1), $r = a \sin 3\theta$

$\therefore$ $\frac{dr}{d\theta} = 3a \cos 3\theta$ $\therefore$ $\tan \phi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{1}{3} \tan 3\theta$

NOTES

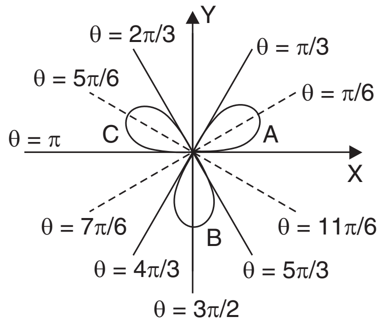
$$\tan\phi = \infty \left( i.e., \quad \phi = \frac{\pi}{2} \right)$$

for $$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}$$

∴ Tangent is perpendicular to the radius vector at the points

$$\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{9\pi}{6}, \frac{11\pi}{6}.$$

Thus, the shape of the curve is as shown in the figure.




**Remark.** We have not considered values of θ outside [0, 2π] because $r = a \sin 3\theta$ is periodic.

**Note 1.** $\cos x = \cos(-x) = \cos(2\pi - x) = \cos(4\pi - x) = ......$

Also $\cos\theta = 0$ gives $\theta = (2n + 1)\pi/2$.

**Note 2. The curve r = a sin nθ or r = a cos nθ consists of n or 2n loops according as n is odd or even.**

**Example 7.** *Trace the curve* $r^2 \cos\theta = a^2 \sin 3\theta$.

**Sol.** The equation of the curve is $r^2 \cos\theta = a^2 \sin 3\theta$ ...(1)

1. **Symmetry.** On changing $r$ to $-r$, equation (1) remains unchanged.

∴ The curve is symmetrical about the pole, *i.e.,* the curve is symmetrical in opposite quadrants.

2. **Origin or Pole.**

Putting $r = 0$ in (1), we get $\sin 3\theta = 0$

∴ $3\theta = n\pi$ when $n$ is zero or any other integer.

∴ $$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{6\pi}{3}$$

∴ The curve passes through the pole and the tangents at the pole are

$$\theta = 0, \theta = \frac{\pi}{3}, \theta = \frac{2\pi}{3}, \theta = \frac{3\pi}{3}, \theta = \frac{4\pi}{3}, \theta = \frac{5\pi}{3}, \theta = \frac{6\pi}{3}.$$

3. **Asymptotes.** Putting $r = \frac{1}{u}$, from (1),

$$u^2 = \frac{\cos\theta}{a^2 \sin 3\theta} \qquad ...(2)$$

Let $u \to 0$. Therefore, $\cos\theta \to 0$, *i.e.,* $\theta \to \frac{\pi}{2}, \frac{3\pi}{2}$

Diff. (2) w.r.t. θ, $$2u\frac{du}{d\theta} = \frac{1}{a^2}\left[\frac{-\sin 3\theta \sin\theta - \cos\theta\,(3\cos 3\theta)}{\sin^2 3\theta}\right]$$

∴ $$\frac{du}{d\theta} = -\frac{(\sin 3\theta \sin\theta + 3\cos\theta \cos 3\theta)}{2a^2 u \sin^2 3\theta}$$

For $\theta_1 = \frac{\pi}{2}$, $$p = \underset{\substack{\theta \to \theta_1 \\ u \to 0}}{\text{Lt}} \left(-\frac{d\theta}{du}\right) = \underset{\substack{\theta \to \pi/2 \\ u \to 0}}{\text{Lt}} \frac{2ua^2 \sin^2 3\theta}{\sin 3\theta \sin\theta + 3\cos\theta\cos 3\theta} = 0$$

∴ The asymptote is $p = r\sin(\theta_1 - \theta)$

or $$0 = r\sin\left(\frac{\pi}{2} - \theta\right) \quad i.e., \quad r\cos\theta = 0 \quad \text{or} \quad x = 0$$

[∵ $x = r\cos\theta$]

*i.e.,* $y$-axis is an asymptote to the curve.

Again, for $\theta_1 = \frac{3\pi}{2}$, $y$-axis is an asymptote.

4. **Points of Intersection.** When $\theta = 0$ ; From (1), $r = 0$ (Also see step 2).



When $\theta = \frac{\pi}{6}$, $\quad r^2 = \frac{2a}{\sqrt{3}} \quad \therefore \quad r = \pm\sqrt{\frac{2a}{\sqrt{3}}}$

$\therefore$ The two points are $\left(\pm\sqrt{\frac{2a}{\sqrt{3}}}, \frac{\pi}{6}\right)$

when $\quad \theta = \frac{\pi}{3}$, $r = 0$ again.

5. **Region w.r.t. tangents at the pole**

From (1), $\quad r^2 = \frac{a^2 \sin 3\theta}{\cos\theta}$

when $\quad 0 \le \theta \le \frac{\pi}{3}$, $r^2$ is + ve $\quad \therefore \quad r$ is real.

when $\quad \frac{\pi}{3} \le \theta \le \frac{\pi}{2}$, $r^2$ is – ve $\quad \therefore \quad r$ is imaginary.

$\therefore$ No portion of the curve lies between the two half-rays $\theta = \frac{\pi}{3}$ and $\theta = \frac{\pi}{2}$.

when $\quad \frac{\pi}{2} \le \theta \le \frac{2\pi}{3}$ $r^2$ is + ve $\quad \therefore \quad r$ is real

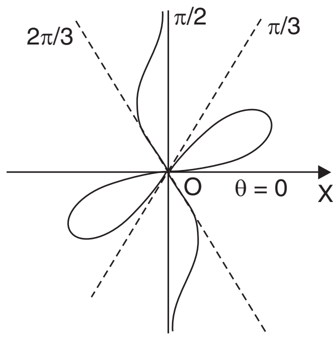
when $\quad \frac{2\pi}{3} \le \theta \le \pi$ $r^2$ is – ve

$\therefore$ $r$ is imaginary.

$\therefore$ No portion of the curve lies between the half rays

$$\theta = \frac{2\pi}{3} \quad \text{and} \quad \theta = \pi.$$

Hence the shape of the curve is as shown.




**Note.** We have not discussed the values of $\theta$ between $\pi$ and $2\pi$ because the curve is symmetrical in opposite quadrants.

**Example 8.** *Trace the curve* $x^5 + y^5 = 5a^2x^2y$.

**Sol.** The equation of the curve is $x^5 + y^5 = 5a^2x^2y$ ...(1)

1. **Symmetry.** The curve is symmetrical in opposite quadrants because equation (1) remains unchanged when both $x$ and $y$ are changed to $-x$ and $-y$ respectively.

2. **Origin.** The curve passes through the origin. The tangents at the origin are $x^2y = 0$ *i.e.,* $x = 0, x = 0, y = 0$.

$\therefore$ The $y$-axis is a cuspidal tangent.

In fact, the origin is a node as well as a cusp.

3. **Asymptotes.** Since co-efficients of $x^5$ and $y^5$ are constants, therefore, the curve has no asymptotes parallel to axes.

Let us now find oblique asymptotes :

From (1), $\phi_5(m) = 1 + m^5$ [Putting $x = 1$ and $y = m$ in fifth degree terms]

$\phi_4(m) = 0$ [$\because$ There are no fourth degree terms in equation (1)]

Putting $\phi_5(m) = 0$, we have $m^5 + 1 = 0$ or $m^5 = -1$ $\therefore$ $m = -1$.

To find $c$, $c\phi_5'(m) + \phi_4(m) = 0$

or $c(5m^4) = 0$ or $c = \dfrac{0}{5m^4} = 0$.

NOTES

$\therefore$ Equation of the asymptote is $y = mx + c$ or $y = -x$.

*Position of the curve w.r.t. the asymptote.* In the second quadrant, $x$ is negative and $y$ is positive.

$\therefore$ R.H.S. of (1) is + ve.

$\therefore$ L.H.S. of (1) namely $y^5 + x^5$ must also be + ve.

Hence $y$ should be numerically $> x$.

Thus, the curve lies above the asymptote $y = -x$.

Because of symmetry in opposite quadrants, the curve approaches the other end (in the fourth quadrant) from below.

4. **Points of Intersections**

(*i*) *With x-axis.* Putting $y = 0$ in (1), $x^5 = 0$ or $x = 0$.

$\therefore$ Point of intersection is (0, 0)

(*ii*) Similarly, point of intersection with $y$-axis is also (0, 0).

(*iii*) *Intersections with the line $y = x$.* Putting $y = x$ in (1), we have

$$2x^5 = 5a^2x^3. \quad \therefore \quad x = 0 \quad \text{or} \quad x = \pm\sqrt{\frac{5}{2}}\,.\,a$$

$$\therefore \qquad y = 0 \quad \text{and} \quad x = \pm\sqrt{\frac{5}{2}}\,.\,a$$

$\therefore$ Point of intersection are (0, 0) ; $\left(\pm\sqrt{\dfrac{5}{2}}\,a, \pm\sqrt{\dfrac{5}{2}}\,a\right)$.

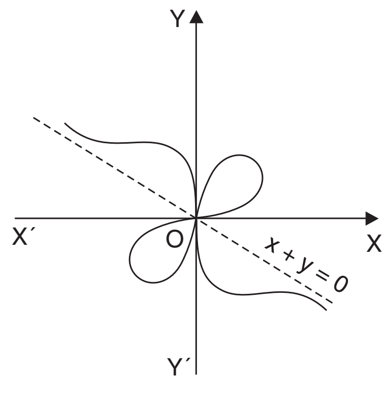
(*iv*) *Intersections with the asymptote $y = -x$.* Solving (1) and $y = -x$, we again have (0, 0) as their intersection.

5. **Region.** Transforming to polars [by putting $x = r\cos\theta$ and $y = r\sin\theta$ in (1), we have]

$$r^2 = \frac{5a^2\cos^2\theta\sin\theta}{\cos^5\theta + \sin^5\theta}.$$

For values of $\theta$ between $\dfrac{3\pi}{4}$ and $\pi$, $r^2$ is negative and so $r$ is Imaginary.

Thus, no portion of the curve lies between the lines $\theta = \dfrac{3\pi}{4}$ and $\theta = \pi$. Thus, the shape of the curve is as shown in the figure.




**Note.** We could do without this step 5 in the above solution [because the conclusion draw in step (5) was already obtained in step (3)].

**NOTES**

## EXERCISE B

*Trace the following curves :*

**1.** $r = a \cos 3\theta$. **2.** $r = a \cos 5\theta$.

**3.** $r = \dfrac{a\theta^2}{1+\theta^2}$.

[**Hint.** $r = a$ is a circular asymptote for this curve $\because \underset{\theta \to \infty}{\text{Lt}} \dfrac{a\theta^2}{1+\theta^2} = a.$]

**4.** (*a*) $r = ae^{m\theta}$, where $a$, $m$ are + ve. (*b*) $r = ae^{\theta \cot \alpha}$.

[**Hint.** Curve in part (*b*) in the same as in (*a*) where $m = \cot \alpha$.]

**5.** $r = a\theta$. **6.** $r^2 \cos 2\theta = a^2$.

[**Hint.** Cartesian equation is $x^2 - y^2 = a^2$.]

**7.** $r = a\,(1 + \sec \theta)$. **8.** $x^4 + y^4 = 4a^2xy$.

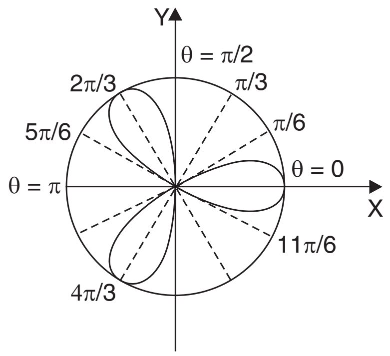
[**Hint.** No portion of the curve lies in second and fourth quadrants.]

**9.** $y^4 - x^4 + xy = 0$. **10.** $x^5 + y^5 = 5ax^2y^2$. **11.** $x^4 + y^4 = 4axy^2$.

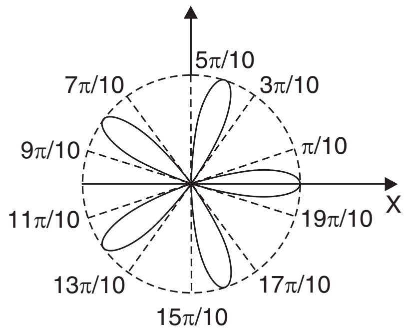
**12.** $\dfrac{2a}{r} = 1 + \cos \theta$ (Parabola).

### Answers

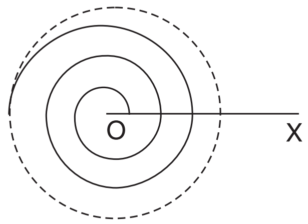
**1.**



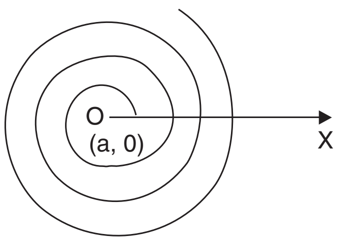

**2.**




**3.**

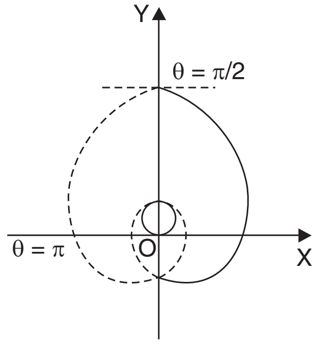



**4.**

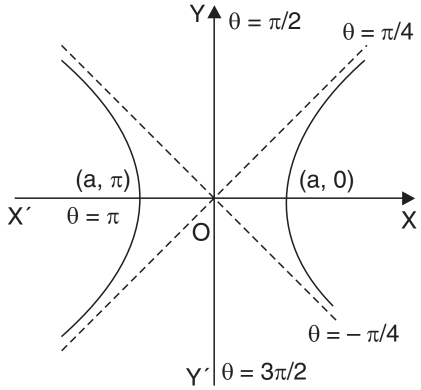



Circular Asymptote $r = a$ is shown as dotted. The part of the curve for negative values is its reflection in the initial line.

**5.**

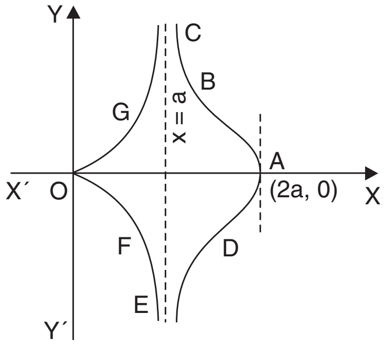



**6.**



NOTES

7.

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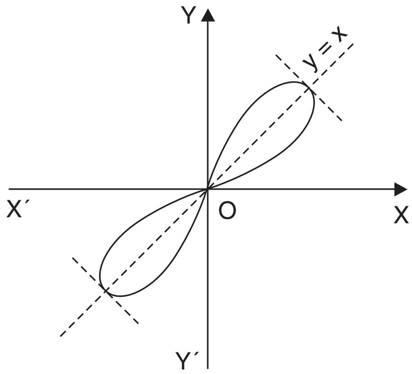


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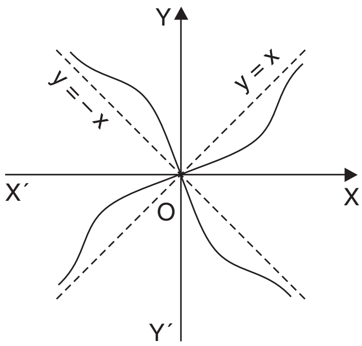
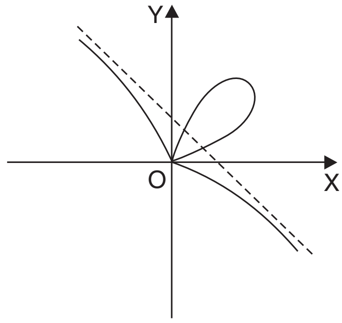
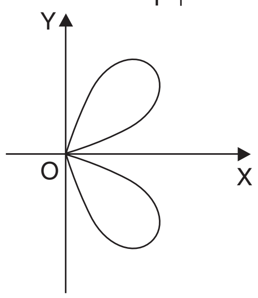


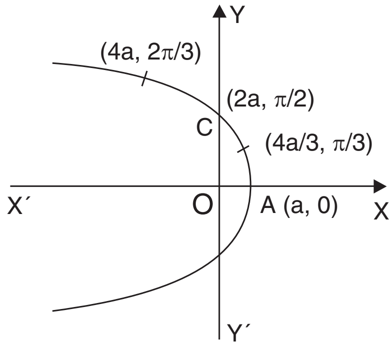

10.




11.




12.




## TRACING OF PARAMETRIC EQUATIONS

Let $x = f(t)$, $y = \phi(t)$, where $t$ is the parameter in the equations of the curve.

**Case I.** If conveniently possible, the parameter is eliminated and the corresponding cartesian equation obtained and then the procedure explained earlier in Art. 2. is followed.

The following results may prove helpful :

**1.** $x = a \cos t, \qquad y = b \sin t.$ (*Ellipse*)

$\therefore \qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1.$

**2.** $x = a \cos t, \qquad y = a \sin t.$ (*Circle*)

$\therefore \qquad x^2 + y^2 = a^2 (\cos^2 t + \sin^2 t) = a^2$

**3.** $x = a\cos^3 t, \quad y = b\sin^3 t$ (*Hypo-cycloid*)

$\therefore \quad \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1.$

**4.** $x = a\cos^3 t, \quad y = a\sin^3 t$ (*Astroid*)

$x^{2/3} + y^{2/3} = a^{2/3}.$

(This curve, we have already traced as example 7)

**5.** $x = t^2, \quad y = t - \frac{1}{3}t^3$

$\therefore \quad y = t(1 - \frac{1}{3}t^2) = t(1 - x/3)$

$\therefore \quad y^2 = t^2(1 - x/3)^2 = x(1 - x/3)^2.$

**6.** $x = a\sin^2 t, \quad y = a\frac{\sin^3 t}{\cos t}$ (*Cissoid*)

$\therefore \quad y = \frac{\sin t}{\cos t}x \quad \therefore \quad y^2 = \frac{\sin^2 t}{1 - \sin^2 t}.x^2 = \frac{x/a \,.\, x^2}{1 - x/a}$

or $\quad y^2(a - x) = x^3.$

**7.** $x = \frac{1 - t^2}{1 + t^2}, \; y = \frac{2t}{1 + t^2}$ (*Circle*)

$\therefore \quad x^2 + y^2 = \frac{(1 - t^2)^2 + 4t^2}{(1 + t^2)^2} = \frac{(1 + t^2)^2}{(1 + t^2)^2} = 1.$

**8.** $x = \frac{3at}{1 + t^3}, \; y = \frac{3at^2}{1 + t^3}.$ (*Folium of Descarte's*)

$x^3 + y^3 = 27a^3t^3\frac{(1 + t^3)}{(1 + t^3)^3} = \frac{27a^3t^3}{(1 + t^3)^2} = 3axy.$

**Case II.** If it is not conveniently possible to eliminate the parameter, then the following procedure is followed :

## Symmetry

(*i*) If on changing $t$ to $-t$ or ($t$ to $\pi - t$) ; $x(= f(t))$ remains unchanged and $y$ $(= \phi(t))$ changes to $-y$ ; then the curve is **symmetrical about x-axis.**

*e.g.,* The parabola $x = at^2$, $y = 2at$ is symmetrical about $x$-axis.

(*ii*) Similarly, if on changing $t$ to $-t$ (or $t$ to $\pi - t$), $x$ changes to $-x$ and $y$ remains unchanged, then the curve is **symmetrical about y-axis.**

*e.g.,* (1) the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ is symmetrical about $y$-axis.

(2) The ellipse $x = a\cos t$, $y = b\sin t$ is symmetrical about $y$-axis.

(changing $t$ to $\pi - t$)

(*iii*) If on changing $t$ to $-t$ ; $x$ changes to $-x$ and $y$ to $-y$, the curve is symmetrical in *opposite quadrants.*

*e.g.,* the rectangular hyperbola $x = ct$, $y = \frac{c}{t}$ is symmetrical in opposite quadrants.

## Origin

If on putting $x = 0$ (or $y = 0$) a real value of $t$ can be found out, which makes $y = 0$ (or $x = 0$), then the curve passes through the origin.

or Put both $x$ and $y$ equal to zero and find values of $t$. If there is any common value of $t$, then the curve passes through the origin.

### Asymptotes

Find asymptotes if any.

NOTES

## GENERAL METHOD TO FIND ASYMPTOTES

### Asymptotes || to x-axis

Find the definite values $d_1$, $d_2$, ...... to which $y$ tends as $x \to +\infty$ or $-\infty$, then $y = d_1$, $y = d_2$ ...... are asymptotes || to $x$-axis.

### Asymptotes || to y-axis

Find the definite values $k_1$, $k_2$, $k_3$, ...... to which $x$ tends as $y \to +\infty$ or $-\infty$, then $x = k_1$, $x = k_2$, ...... are asymptotes || to $y$-axis.

### Oblique asymptotes

$y = mx + c$ is an oblique asymptote to a curve where

$$m = \underset{\substack{x \to \infty \\ y \to \infty}}{\text{Lt}} \frac{y}{x} \quad \text{and} \quad c = \underset{\substack{x \to \infty \\ y \to \infty}}{\text{Lt}} (y - mx).$$

### Points of Intersection

(*i*) The points of intersection of the curve with $x$-axis are given by the roots of the equation $y = \phi(t) = 0$.

(*ii*) The points of intersection of the curve with $y$-axis are given by the roots of the equation $x = f(t) = 0$.

### Region

If easily possible, find the greatest and least values of $x$ and $y$ and therefore, the lines parallel to the axes between which the curve lies or does not lie.

**For example.** If the curve is $x = a \cos \theta$, $y = b \sin \theta$ ; then the greatest value of $x$ is $a$ and the least value is $-a$.

($\because$ greatest and least values of $\cos \theta$ are 1 and $-1$)

$\therefore$ The curve lies entirely between the lines $x = \pm a$.

Similarly, the curve lies entirely between the lines $y = \pm b$.

### Special points

Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and then consider the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in suitable intervals of $t$.

Find $\frac{dy}{dx}\left(= \frac{dy/dt}{dx/dt}\right)$.

Give to the parameter $t$ certain values and find the corresponding values of $x$, $y$ and the slope of the tangent namely $\frac{dy}{dx}$. Plot these points (whose cartesian co-ordinates are known to us). Find those points on the curve for which $\frac{dy}{dx} = 0$ or $\infty$. Also find $\frac{d^2y}{dx^2}$ and discuss for concavity and points of inflexion (as explained in Step 6 of Art. 2).

NOTES

## SOLVED EXAMPLES

**Example 9.** *Trace the cycloid* $x = a\,(\theta + \sin\theta)$. $y = a(1 + \cos\theta)$.

**Sol.** The parametric equations of the cycloid are

$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta) \qquad \ldots(1)$$

Let us trace this curve firstly for values of θ in the interval $[-\pi, \pi]$.

1. **Symmetry.** On changing θ to $-\theta$ in (1), changes to $-x$ and $y$ remains unchanged.

∴ The curve is symmetrical about $y$-axis.

2. **Origin.** Putting $y = 0$, $a(1 + \cos\theta) = 0$

$$\therefore \qquad \cos\theta = -1 \quad \text{or} \quad \theta = \pi$$

For $\theta = \pi$, $\qquad x = a(\theta + \sin\theta) = a\pi \neq 0$

∴ The curve does not pass through the origin.

3. **Asymptotes.** The curve has no asymptotes.

4. **Points or Intersection**

(*i*) *Intersections with x-axis.* Putting $y = 0$ in (1), we get $\theta = \pi$ which gives $x = a\pi$.

∴ Intersection with $x$-axis is $(a\pi, 0)$.

(*ii*) *Intersection with y-axis.* Putting $x = 0$ in (1), we have $\theta + \sin\theta = 0$ which is satisfied by only $\theta = 0$ and for $\theta = 0$, $y = a\,(1 + \cos\theta) = 2a$.

∴ Intersection with $y$-axis is $(0, 2a)$.

5. **Region.** We know that $-1 \leq \cos\theta \leq 1$

$$\therefore \qquad 1 - 1 \leq 1 + \cos\theta \leq 1 + 1$$

or

$$0 \leq (1 + \cos\theta) \leq 2$$

$$\therefore \qquad 0 \leq a(1 + \cos\theta) \leq 2a \quad \text{or} \quad 0 \leq y \leq 2a$$

∴ Curve lies entirely between the lines $y = 0$ and $y = 2a$.

6. **Special points.** From (1),

$$\frac{dx}{d\theta} = a\,(1 + \cos\theta) \text{ and } \frac{dy}{d\theta} = -a\sin\theta$$

$$\because \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\sin\theta}{a\,(1+\cos\theta)} = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = -\tan\frac{\theta}{2} \qquad \ldots(2)$$

Corresponding values of $x$, $y$ and $dy/dx$ for different values of θ are given below

| | θ | $-\pi$ | $-\pi/2$ | 0 | $\pi/2$ | $\pi$ |
|---|---|---|---|---|---|---|
| From (1), | $x$ | $-a\pi$ | $-a\left(\frac{\pi}{2}+1\right)$ | 0 | $a\left(\frac{\pi}{2}+1\right)$ | $a\pi$ |
| | $y$ | 0 | $a$ | $2a$ | $a$ | 0 |
| From (2), | $\frac{dy}{dx}$ | $\infty$ | 1 | 0 | $-1$ | $-\infty$ |

∴ Points on the curve are

$$(-a\pi, 0), \left[-a\left(\frac{\pi}{2}+1\right), a\right], [0, 2a], \left[a\left(\frac{\pi}{2}+1\right), a\right], [a\pi, 0].$$

Tangents at the points $[-a\pi, 0]$ and $[a\pi, 0)$ are parallel to $y$-axis.

$$(\because \quad dy/dx = \infty \text{ or } -\infty)$$

Tangent at the point (0, 2$a$) is || to $x$-axis $(\because \; dy/dx = 0)$

Form (2), $$\frac{dy}{dx} = -\tan\frac{\theta}{2}$$

NOTES

Again diff. w.r.t. $x$, $$\frac{d^2y}{dx^2} = -\left(\sec^2\frac{\theta}{2}\right)\frac{1}{2}\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{2}\sec^2\frac{\theta}{2}\cdot\frac{1}{2a}\sec^2\frac{\theta}{2}$$

$$\because \frac{d\theta}{dx} = \frac{1}{a(1+\cos\theta)} = \frac{1}{2a\cos^2\theta/2} = \frac{1}{2a}\sec^2\frac{\theta}{2}$$

or $$\frac{d^2y}{dx^2} = \frac{-1}{4a}\sec^4\frac{\theta}{2}$$

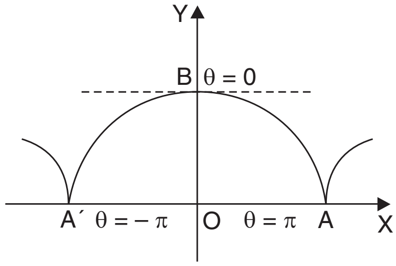
$\therefore$ $\frac{d^2y}{dx^2}$ is negative for all values of θ.

$\therefore$ The curve is concave downwards.

*For values of θ > π or < – π, the same types of branches of the curve will be obtained.*

The curve consists of congruent **arches** on both sides of $y$-axis which extend to infinity.

Hence the shape of the curve is as shown in the figure.




**Note.** B is called **vertex** of the cycloid and Line A′A *i.e.,* the line joining the two end points is called **base** of the cycloid and $a$ is called the radius of the generating circle.

**Remark.** We have taken θ from – π to π to get one **arch** of the cycloid or one complete cycloid. We will again take θ from – π to π for the cycloid

$$x = a(\theta + \sin\theta),\; y = a(1 - \cos\theta)$$

*i.e.,* **We take θ in [– π, π] for the cycloids**

$$\mathbf{x = a(\theta + \sin\theta),\; y = a(1 \pm \cos\theta)}$$

**We will take θ in [0, 2π] for the cycloids**

$$\mathbf{x = a(\theta - \sin\theta),\; y = a(1 \pm \cos\theta)}$$

($\because$ For these two cycloids, for values of θ < 0 or θ > 2π, the same types of branches of the curve will be obtained.)

**Example 10.** *Trace the curve*

$$x = a\left[\cos t + \frac{1}{2}\log\tan^2\frac{t}{2}\right],\; y = a\sin t. \qquad \textbf{(Tractrix)}$$

**Sol.** The equations of the curve are

$$x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2},\; y = a\sin t \qquad \ldots(1)$$

1. **Symmetry.** (*i*) On changing $t$ to – $t$ in (1), $x$ remains unchanged and $y$ changes to – $y$.

$\therefore$ The curve is symmetrical about $x$-axis.

(*ii*) On changing $t$ to π – $t$ in (1), $y$ is unchanged and $x$ changes to – $x$.

$\therefore$ Symmetry about $y$-axis.

NOTES

2. **Origin.** Putting $y = 0$, we get $\sin t = 0$ or $t = 0$ and then

$$x = a + \frac{a}{2}\log 0 = -\infty.$$

$\therefore$ The curve does not pass through the origin.

3. **Asymptotes.** When $t = 0$, $x \to -\infty$ and $y = 0$.

Thus $y = 0$, *i.e.,* $x$-axis is an asymptote to the curve. (*See step* (3) *Art. 6*)

4. **Points of Intersection.** The curve does not meet $x$-axis.

The curve meets $y$-axis (putting $x = 0$) where

$$x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2} = 0 \text{ which is satisfied by } t = \pm\frac{\pi}{2}.$$

and then $y = a\sin\left(\pm\frac{\pi}{2}\right) = \pm a$. Thus, the curve meets $y$-axis in the points $(0, \pm a)$.

5. **Region.** We know that $-1 \le \sin t \le 1$

$\therefore$ $-a \le a\sin t \le a$ or $-a \le y \le a$

Thus, the curve lies entirely between the lines $y = \pm a$.

6. **Special Points.** (*i*) From (1),

$$\frac{dx}{dt} = -a\sin t + \frac{1}{2}a \cdot \frac{1}{\tan^2 t/2} \cdot 2\tan\frac{t}{2} \cdot \sec^2\frac{t}{2}\cdot\frac{1}{2}$$

$$= a\left(-\sin t + \frac{1}{2\sin t/2\cos t/2}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t} \text{ and } \frac{dy}{dt} = a\cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \tan t \qquad \dots(2)$$

$$\frac{dy}{dx} = \infty, \text{ when } t = \pm\frac{\pi}{2} \text{ and then } x = 0,\ y = \pm a.$$

Thus, at the points $(0, \pm a)$ the tangent is || to $y$-axis.

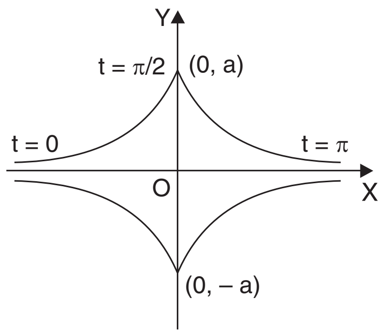
*i.e.,* $y$-axis itself is a tangent at the points $(0, \pm a)$.

$\frac{dy}{dx} = 0$ when $t = 0$ and then $x \to \infty$, $y = 0$, showing that $y = 0$ or $x$-axis is an asymptote as proved earlier.

From (2), $\frac{dy}{dx} = \tan t$

Diff. w.r.t. $x$,

$$\frac{d^2y}{dx^2} = \sec^2 t\,\frac{dt}{dx} = \sec^2 t \cdot \frac{\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t}$$




$\therefore$ $\frac{d^2y}{dx^2}$ is + ve for values of $t$ in $[0, \pi]$.

$\therefore$ The curve is concave upwards for values of $t$ in $[0, \pi]$.

The shape of the curve is as shown in the figure.

**Example 11.** *Trace the curve* $x = \dfrac{a(t+t^3)}{1+t^4}, y = \dfrac{a(t-t^3)}{1+t^4}$.

NOTES

**Sol.** Equations of the curve are $x = \dfrac{a(t+t^3)}{1+t^4} = \dfrac{at\,(1+t^2)}{1+t^4}$ ...(1)

and $y = \dfrac{a(t-t^3)}{1+t^4} = \dfrac{at\,(1-t^2)}{1+t^4}$ ...(2)

Let us eliminate $t$ from Equations (1) and (2) to form Cartesian Equation.

Dividing Eqn. (1) by Eqn. (2), we have

$$\frac{x}{y} = \frac{1+t^2}{1-t^2} \quad \text{or} \quad y + t^2y = x - t^2x \quad \text{or} \quad t^2(x+y) = x-y$$

$\therefore$ $t^2 = \dfrac{x-y}{x+y}$ ...(3)

S.B.S. of Eqn. (1) and cross-multiplying $x^2(1+t^4)^2 = a^2t^2\,(1+t^2)^2$

Putting the value of $t^2$ from (3),

$$x^2\left[1+\frac{(x-y)^2}{(x+y)^2}\right]^2 = \frac{a^2\,(x-y)}{x+y}\left(1+\frac{x-y}{x+y}\right)^2$$

or $$x^2\,\frac{[(x+y)^2+(x-y)^2]^2}{(x+y)^4} = \frac{a^2\,(x-y)}{(x+y)}\left[\frac{2x}{(x+y)}\right]^2$$

or $$\frac{4x^2\,(x^2+y^2)^2}{(x+y)^4} = \frac{4a^2x^2(x-y)}{(x+y)^3}$$

Dividing both sides by $\dfrac{4x^2}{(x+y)^3}$, $\dfrac{(x^2+y^2)^2}{x+y} = a^2(x-y)$

or cross-multiplying $(x^2+y^2)^2 = a^2(x^2-y^2)$ ...(4)

**1. Symmetry.** Curve (4) is symmetrical both about $x$-axis and $y$-axis

[$\because$ It contains only *even* powers of $y$ and only *even* powers of $x$].

**2. Origin.** Curve (4) passes through the origin.

Tangents at the origin are $a^2(x^2-y^2) = 0$ (Equating lowest degree terms to zero)

But $a \neq 0$ $\therefore$ $x^2 - y^2 = 0$ or $y^2 = x^2$ $\therefore$ $y = \pm x$.

**3. Asymptotes.** The curve has no asymptotes.

[$\because$ Coeff. of $x^4$ is $1 \neq 0$ ; Coeff. of $y^4$ is $1 \neq 0$ ;

$\phi_4(m) = (1+m^2)^2 = 0$ gives no real values of $m$]

**4. Points of Intersection**

(*i*) **Intersections with x-axis**

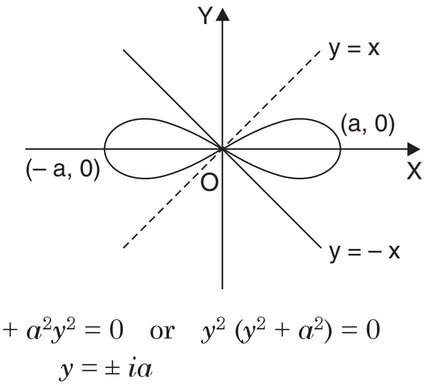
Putting $y = 0$ in Eqn. (4), $x^4 = a^2x^2$

or $x^4 - a^2x^2 = 0$ or $x^2\,(x^2 - a^2) = 0$

$\therefore$ Either $x = 0$ or $x^2 = a^2$

*i.e.,* $x = \pm a$

$\therefore$ Intersections of the curve (4) with $x$-axis are $(-a, 0)$ $(0, 0)$ and $(a, 0)$.




(*ii*) **Intersections with y-axis**

Putting $x = 0$ in Eqn. (4), $y^4 = -a^2y^2$ or $y^4 + a^2y^2 = 0$ or $y^2\,(y^2 + a^2) = 0$

$\therefore$ $y = 0$ or $y = \pm ia$

$\therefore$ The only point of intersection of curve (4) with $y$-axis is (0, 0).

Thus the shape of the curve is as shown in the figure.

## EXERCISE C

*Trace the following curves :*

**1.** $x = a\,(\theta + \sin\theta),\ y = a\,(1 - \cos\theta)$.
[**Hint.** See Remark Example 1.]

**2.** $x = a\,(\theta - \sin\theta),\ y = a\,(1 + \cos\theta)$.
[**Hint.** See Remark Example 1.]

**3.** $x = a\,(\theta - \sin\theta),\ y = a\,(1 - \cos\theta)$

**4.** $x = t^2,\ y = t - \frac{1}{3}\,t^3$.

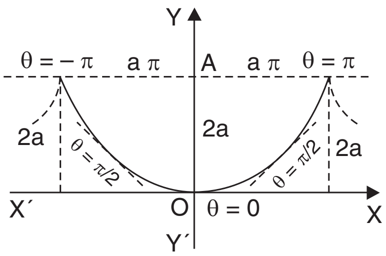
**5.** $x = \dfrac{a\,(1-t^2)}{1+t^2},\quad y = \dfrac{at\,(1-t^2)}{1+t^2}$ [**Hint.** Eliminate $t$ by dividing.]

**6.** $x = a\,[\cos\theta - \log\,(1 + \cos\theta)],\ y = a\sin\theta$.

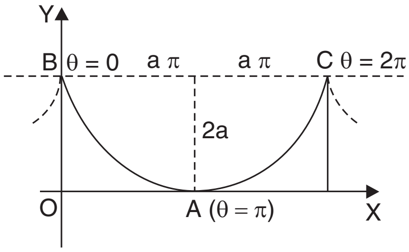
**7.** $x = a\sin 2\theta\,(1 + \cos 2\theta),\ y = a\cos 2\theta\,(1 - \cos 2\theta)$.

### Answers

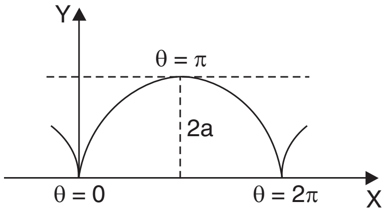
**1.**



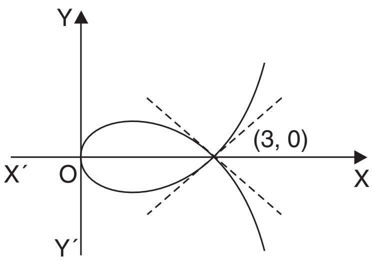

**2.**



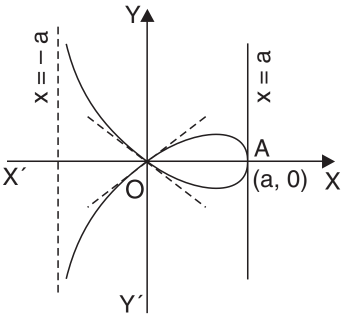

**3.**



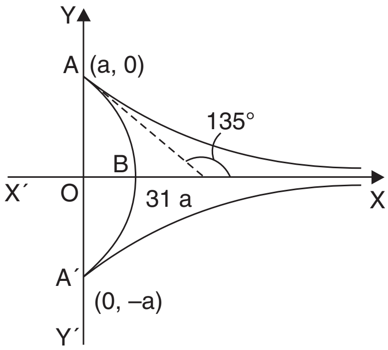

**4.**




**5.**




**6.**




**7.**